Brandenburgische Technische Universität Cottbus - Senftenberg

This cannot be the end!

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joint work with
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How to start to find an end?

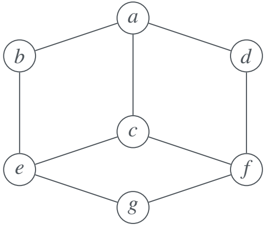
BFS — This cannot be the end(-vertex)!

Idea of BFS:

initialize queue $Q = \{s\}$
iteratively take top vertex v from Q
add all non-queue neighbors of v to end of Q

Q = $\{$s$\}$; i=1;
foreach $v \in Q$ **do**
  $\sigma(i) \leftarrow v$; i++;
  foreach *unvisited neighbor* w *of* v *with* $w \notin Q$ **do**
    append w to Q
  end
end

Example

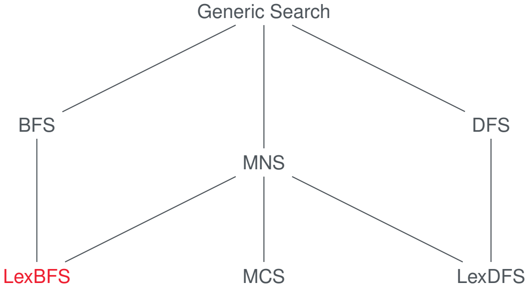


- Can d be endvertex of a BFS?
  Yes! Any BFS starting at e ends at d
- Can c be endvertex of a BFS?
  No! all non-neighbors of c
  have distance 2 from c but excentricity 3

What is so great about ends?

- end-vertex of BFS: helpful for fast diameter computation
- end-vertex of LexBFS: is simplicial in chordal graphs
  key property for recognition and many opt. alg. in chordal graphs
- end-vertex of LexBFS in cocomparability graphs: always a source/sink in some transitive orientation
- end-vertex of LDFS in cocomparability graphs: start-vertex of hamiltonian path (if it exists)
- end-vertex of LexBFS in AT-free graphs: dominating pair vertex
- end-vertices of graph searches in general: their properties are the key for many multi-sweep algorithms on graphs

On the search for the right search



LexBFS — This cannot be the end(-vertex)!

Idea of LexBFS:

iteratively select vertex with lexicogr. largest label; selected vertex appends number $(n-i)$ to label of neighbors

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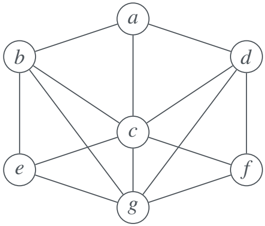
foreach $v \in V$ **do** label(v) = $\emptyset$;
label(s) = $\{0\}$; $n = |V|$
for $i \leftarrow 1$ **to** n **do**
  $v \leftarrow$ unnumbered vertex with lexic. largest label $l(v)$;
  $\sigma(i) \leftarrow v$;
  foreach *unnumb. neighbor w of v* **do**
    append $(n-i)$ to $l(w)$
  end
end

Theorem (Corneil, K., Lanlignel, 2010)

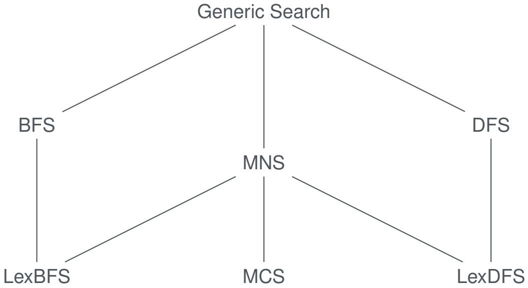
It is NP-hard to decide whether a given vertex is end-vertex of some LexBFS.

Example



- g is end-vertex of a BFS: acbdefg
- Can g be end-vertex of a LexBFS?
  No!
  The only non-neighbor of g is a, thus have to start LexBFS at a
  However, by LeXBFS selection rule g will be visited before e and f

What makes this problem hard?



BFS — This cannot be the end(-vertex)!

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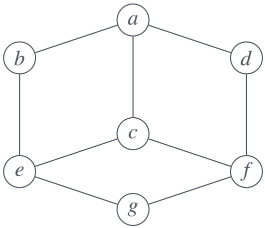
Example



Theorem (Charbit, Habib, Mamcarz '14)

It is NP-hard to decide whether a given vertex is end-vertex of some BFS.

Known results for the end-vertex problem

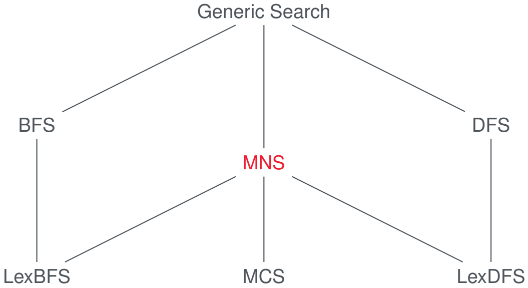
	BFS	LBFS	DFS	LDFS	MCS	MNS	GenS
All Graphs	NPC	NPC	NPC	NPC	?	?	**P**
Weakly Chordal	**NPC**	**NPC**	NPC	**NPC**	?	?	P
Chordal	?	?	NPC	?	?	**P**	P
Interval	?	P	?	?	?	P	P
Split	**P**	**P**	**NPC**	**P**	?	P	P

- Corneil, Köhler, Lanlignel, 2010
- Berry, Blair, Bordat, Simonet, 2010
- Charbit, Habib, Mamcarz, 2014
- Kratsch, Liedloff, Meister 2015: Exact algorithms for BFS and DFS ($O^*(2^n)$); strong connection between end-vertex for DFS and Hamiltonian path problem

Our work:

- maximal neighborhood search (MNS)
- maximum cardinality search (MCS)

On the search for the right search



MNS — Maximal Neighborhood Search

Idea of MNS:

At each step pick: a vertex whose set of neighbors already explored is maximal with respect to set inclusion.

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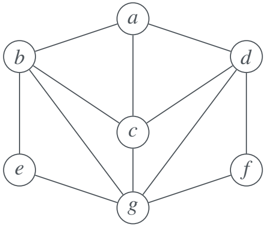
foreach $v \in V$ **do** label(v) = $\emptyset$;
label(s) = $\{0\}$;
for $i \leftarrow 1$ **to** n **do**
  $v \leftarrow$ unnumbered vertex with inclusion maximal label $l(v)$;
  $\sigma(i) \leftarrow v$;
  foreach *unnumb. neighbor w of v* **do**
    append i to $l(w)$
  end
end

→ every LexBFS and every LexDFS is an MNS

→ if G chordal then MNS perfect elim. ordering

Example



resulting MNS ordering: adcgfbe

How Hard is end-vertex for MNS?

Charbit, Habib, Mamcarz: end-vertex problem for MNS open (in their complexity table: ?(P) for this problem)

MNS — How Hard is End-Vertex Problem?

Theorem

The end-vertex problem for MNS is NP-complete.

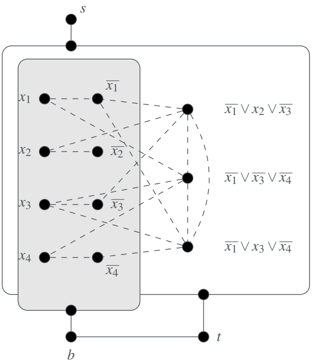
Proof: (reduction from 3-SAT).

let $\mathscr{I}$ be instance of 3-SAT:

- variables of $\mathscr{I}$: $x_1, \dots, x_n$
- clauses of $\mathscr{I}$: $C_1, \dots, C_k$

construct graph $G(\mathscr{I}) = (V, E)$ as follows:

- literal vertices x_i, $\overline{x_i}$: compl. of perf. matching
- clause vertices c_j: independent set
- additional vertices b, s, t
- c_j: adjacent to all literal vertices except "its own"
- b: adjacent to all literal vertices
- s: adjacent to all vertices but b and t
- t: adjacent to all but s



□

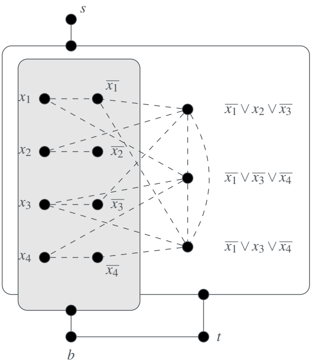
MNS — NP-Completeness Proof

Question

Can t be end-vertex of an MNS?

Observations

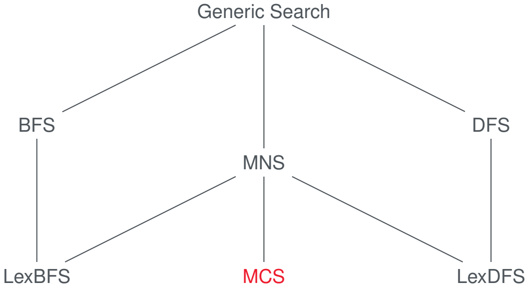
- If MNS chooses a clause vertex before vertex b or b before s, then t cannot be the end-vertex.
- For t being the end-vertex of MNS, the algorithm has to choose s and an assignment right at the beginning.
- For each instance $\mathscr{I}$ of 3-SAT, the graph $G(\mathscr{I})$ is weakly chordal.



Theorem

3-SAT instance has satisfying assignment iff t is end-vertex of an MNS in G

On the search for the right search



→ every MNS for a chordal graph is a perfect elimination ordering

MCS — Maximum Cardinality Search

Idea of MCS:

At each step: pick a vertex whose set of neighbors already explored is maximal with respect to cardinality.

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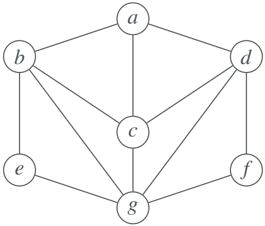
foreach $v \in V$ **do** label(v) = $\emptyset$;
label(s) = $\{0\}$;
for $i \leftarrow 1$ **to** n **do**
    $v \leftarrow$ unnumbered vertex with label $l(v)$ that has maximum cardinality;
    $\sigma(i) \leftarrow v$;
    foreach *unnumb. neighbor* w *of* v **do**
        append i to $l(w)$
    end
end

Example



resulting MCS ordering: adcgbfe

How hard is end-vertex problem of MCS?

Nothing known up to now.

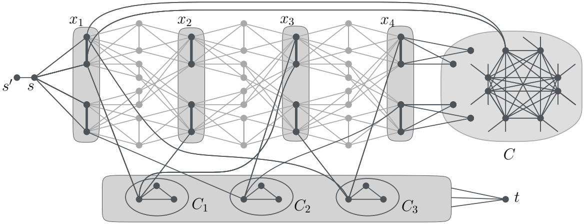
MCS — How Hard is End-Vertex Problem?

Theorem

The end-vertex problem for MCS is NP-complete.

Idea of proof: (reduction from 3-SAT).

- If 3-SAT instance has satisfying assignment then exists MCS ordering with t being end-vertex.



- 3-SAT satisfying assignment iff t is end-vertex of MCS

□

MCS — End-Vertex in Split Graphs

Theorem

$G = (C \cup I, E)$ split graph (C maximal clique, I indep. set).
$t \in V$ is end-vertex of some MCS ordering σ iff (I) t is simplicial and (II) the neighborhoods of the vertices with a smaller degree than t are totally ordered by inclusion.

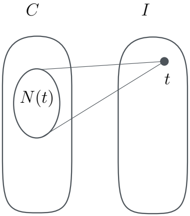
Idea of proof. $\Rightarrow$.

Assume t is the last vertex in σ.

(I) G is split, thus it is chordal. Hence, end-vertex of MCS is simplicial [Tarjan, Yannakakis, 1984]

(II) Assume there exist u, v with smaller degree than t such that $N(u) \not\subseteq N(v)$ and $N(v) \not\subseteq N(u)$.

- can show that $u, v \in I$
- wlog v taken before u in MCS
- $\Rightarrow$ all neighbors of v visited before u
- as $N(u), N(v)$ incomparable: $\exists\, w \in N(v) \setminus N(u)$
- $\Rightarrow$ all vertices of C visited before u
- now label of t larger than label of u, thus t chosen before u
  Contradiction!



$N(u)$

$N(v)$

$\square$

MCS — End-Vertex in Split Graphs

Theorem

$G = (C \cup I, E)$ split graph (C maximal clique, I indep. set).
$t \in V$ is end-vertex of some MCS ordering σ iff (I) t is simplicial and (II) the neighborhoods of the vertices with a smaller degree than t are totally ordered by inclusion.

Idea of proof. $\Leftarrow$.

- t be simplicial and
  neighborhoods of vertices with smaller degree than t totally ordered by set inclusion.
- Define $U := \{u \in V \mid d(u) < d(t)\}$.
- t simplicial $\Rightarrow U \subseteq I$.
- take the neighborhoods of all vertices $u \in U$ in the order of the inclusion ordering.
- Every time the complete neighborhood of a vertex u is taken: take u.
  $\Rightarrow$ all these vertices are taken before t.
- If $t \in C$: then $U = I$ and we can take the remaining vertices of C in an arbitrary ordering, with t being last.
- If $t \in I$: we first take the remaining vertices of C. Since the neighborhood of t is not greater than the neighborhood of all remaining vertices, we can take t as last vertex.

MCS — End-Vertex in (Unit) Interval graphs

Theorem (Gilmore and Hoffman, 1964)

A graph G is an interval graph if and only if the maximal cliques of G can be linearly ordered such that, for every vertex $v \in G$, the maximal cliques containing v occur consecutively.

Lemma

Let $G = (V, E)$ be an interval graph and let $C_1, C_2, \ldots, C_k$ be linear order of the maximal cliques of G. Then $t \in V$ is the last vertex of some MCS-ordering σ of G ***if****:*

1. *t is simplicial, and*
2. *If C_i is the unique clique containing t, then $i = 1$ or $i = k$ or $C_{i-1} \cap C_i \subseteq C_i \cap C_{i+1}$ and $|C_i \cap C_{i+1}| \leq |C_j \cap C_{j+1}|$ for all $j > i$, or the same holds for the reverse order $C_k, C_{k-1}, \ldots, C_1$.*

Theorem

Let $G = (V, E)$ be a unit interval graph and let $C_1, C_2, \ldots, C_k$ be linear order of the maximal cliques of G. Then $t \in V$ is the last vertex of some MCS-ordering σ of G ***iff****:*

1. *t is simplicial, and*
2. *If C_i is the unique clique containing t, then $i = 1$ or $i = k$.*

Conclusions

- Finding an end is surprisingly difficult
- MCS is a quite challenging search for end-vertex problem
- These simple graph searches are not well understood.
- Big step: repeated application of (LBFS) searches (Dusart and Habib)
- Need more and better tools for analysing search algorithms
- A lot of work has to be done here

... and there is no time for retirement!

Thank you Michel and lets keep working!