A Tasty Retrospective for After Lunch

Martin Charles Golumbic
University of Haifa

A celebration with Michel Habib

Graph Sandwich Problems

Guessing and Filling-in Missing Edges

Dealing with Partial Information

missing data and deducing consistency

Take your favorite graph property Π

The Π graph *sandwich problem* asks:

Given: - a vertex set V

- a mandatory edge set E^1
- a larger edge set E^2 $(E^1 \subseteq E^2)$

Is there a (sandwich) graph $G = (V,E)$

with $E^1 \subseteq E \subseteq E^2$ that satisfies property Π ?

Optional edges : $E^2 - E^1$

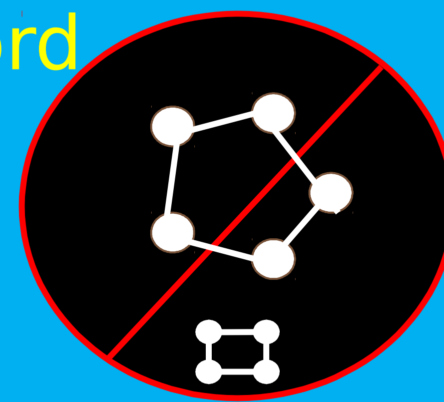
Forbidden edges: $V \times V - E^2$

Remark.
The Classical Recognition Problem is the case where $E^2 = E^1$
nothing optional !

Tasty Examples

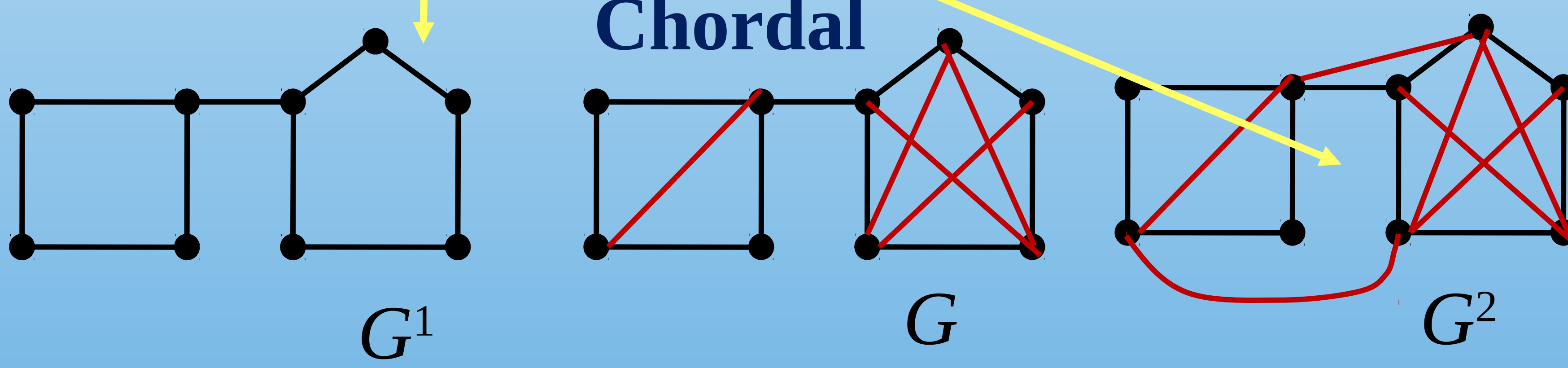
A Chordal Graph Sandwich

chordal graph: every cycle of length $\geq$ 4 has a chord



Cycle C_5

Chordal

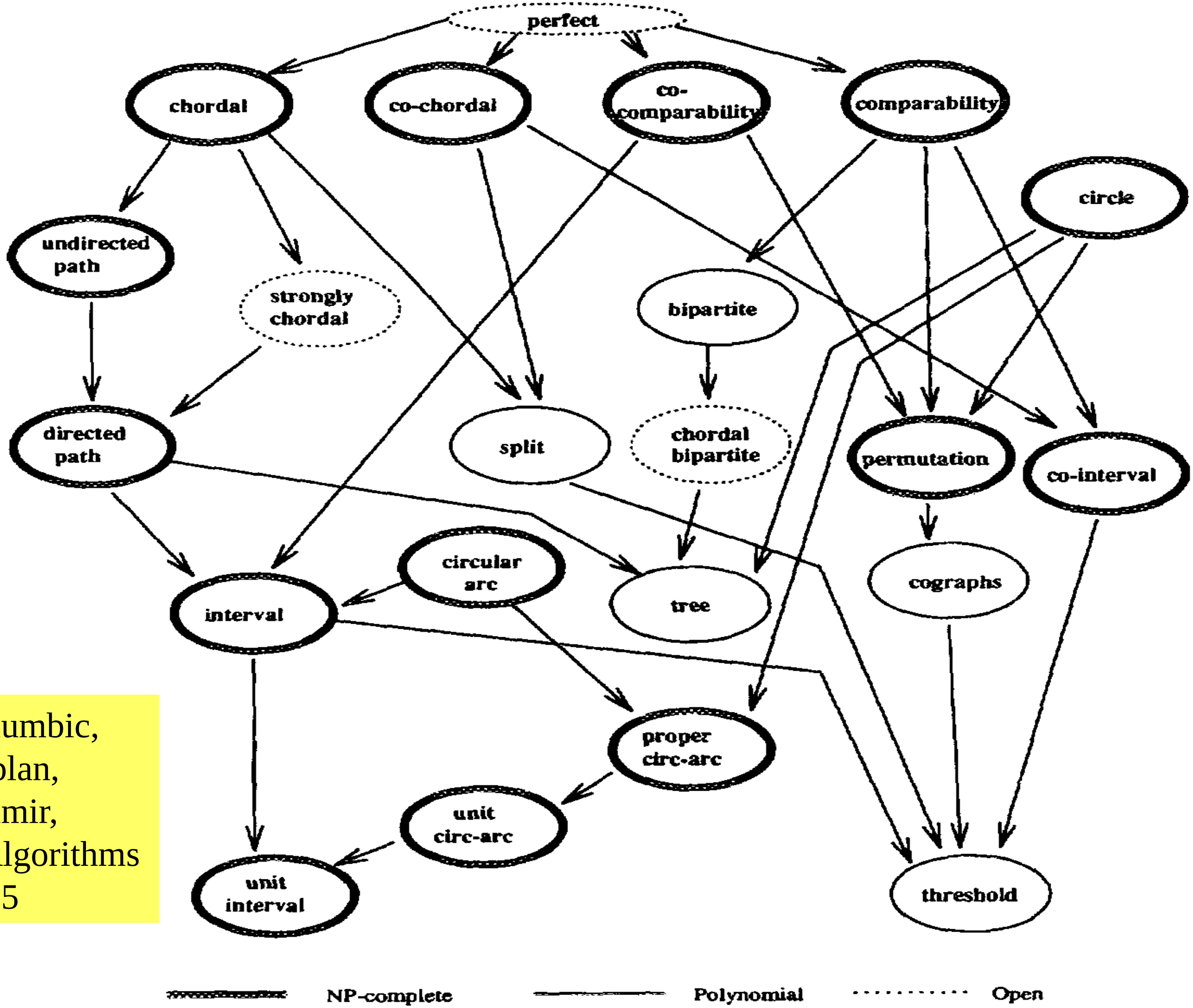


Early references on Graph Sandwich Problems:

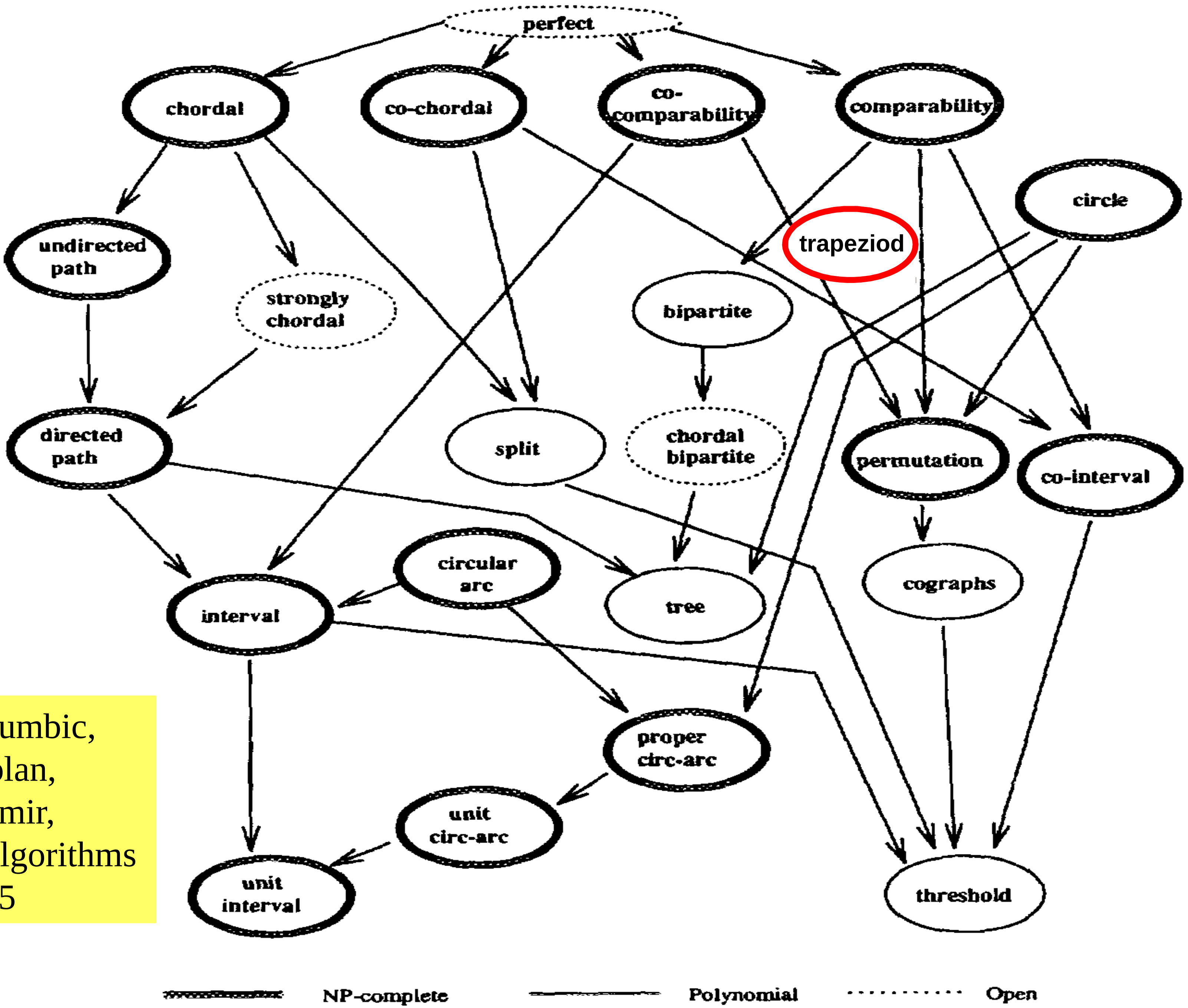
M.C. Golumbic, R. Shamir, J. ACM 1993
Interval Graphs (NP-complete)

H.L. Bodlaender, M.R. Fellows and T.J. Warnow, ICALP 1992
Chordal Graphs (NP-complete)

M.C. Golumbic, H. Kaplan, R. Shamir, J. Algorithms 1995
Permutation, Comparability, Circle, … (NP-complete)
Split, Threshold, Cographs, … (Polynomial)

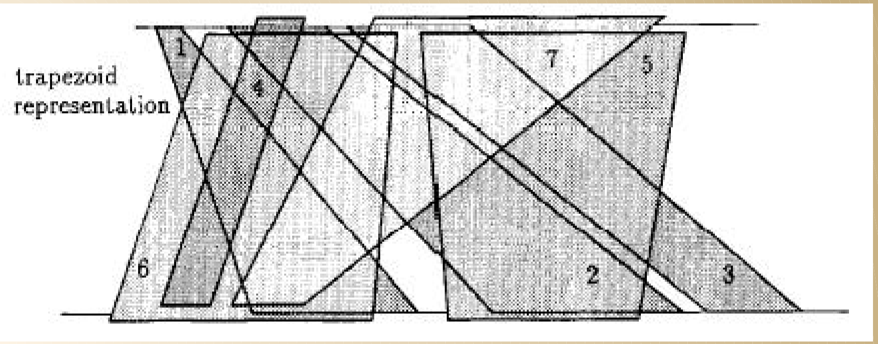


Golumbic,
Kaplan,
Shamir,
J. Algorithms
1995



Golumbic,
Kaplan,
Shamir,
J. Algorithms
1995

Trapeziod Graph Sandwich Problem



- Recognition is Polynomial

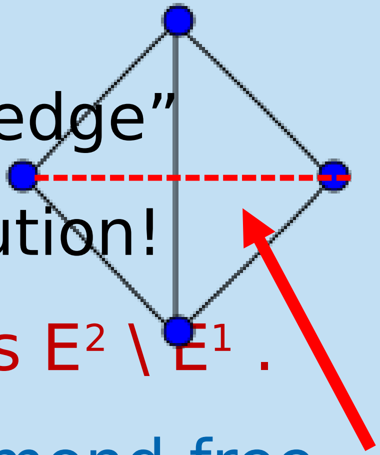
  (Ma and Spinrad [1994], Langley [1995])
- Sandwich is NP-Complete

  You won’t find this in the literature
- Reduction uses the Betweenness Problem

  Similar to the proof for permutation graphs, cocomparability graphs, and unit interval graphs

  Golumbic, Kaplan, Shamir [1995]

Diamond-free Graph Sandwich Problem



If $G^1=(V,E^1)$ has a diamond, then its "non-edge"

will be <u>*forced*</u> to be in any sandwich solution!

So check that it is among the optional edges $E^2 \setminus E^1$.

Then either <u>*ADD IT*</u> to the (potential) diamond-free sandwich,

or FAIL otherwise.

This may force additional edges to be added.

Repeat "forcing of additions"

-- If it stops with no failure, then the resulting filled-in graph will be a diamond-free sandwich graph.

What is your Favorite Property Π ?

EXERCISES:

- **Trees:**

  The Tree Sandwich Problem is Linear

- **Caterpillars:**

  The Caterpillar Sandwich Problem is NP-Complete

- **Planar Graphs:**

  The Planar Graph Sandwich Problem is just Recognition that G^1 is Planar

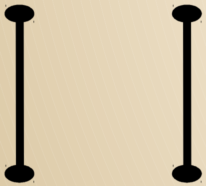
The Chain Graph Sandwich Problem

Simone Dantas, Celina M. H. de Figueiredo,
Martin Golumbic, Sulamita Klein and Frederic Maffray

Annals of Operations Research, 2011.

Definition: A Chain Graph is a $2K_2$ - free bipartite graph.

$2K_2$



The forbidden subgraph characterizing chain graphs.

The chain graph sandwich problem is NP-complete.

This result stands in contrast to

1) **the case where E^1 is a connected graph, (linear-time)**

2) **the threshold graph sandwich problem, (linear-time)**

3) **the chain probe graph problem (polynomial-time)**

The Matrix View of Sandwiches

Adjacency matrix: {0, 1, ✶ } entries, where ✶ means <u>optional</u> -- *don’t know* or *don’t care*.

0						
	0			0 / 1 / ✶		
		0				
			0			
				0		
	0 / 1 / ✶				0	
						0

Matrix Sandwich Problems

- The *consecutive ones* matrix sandwich problem and the *circular ones* matrix sandwich problem are NP-complete. Golumbic and Wassermann [1996]

- The *Ferrers* matrix sandwich problem can be solved in O(*mn*) time. Golumbic [1996]

Matrix Sandwich Problems, cont.

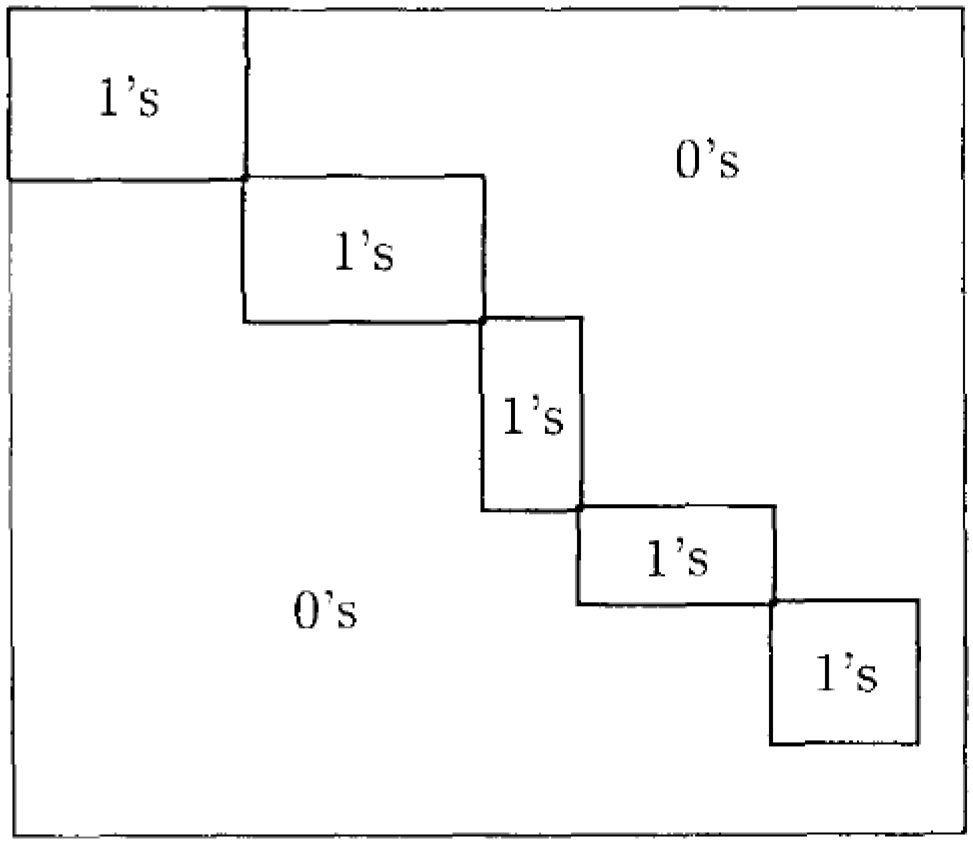


Fig. 1. A rectangular block pattern.

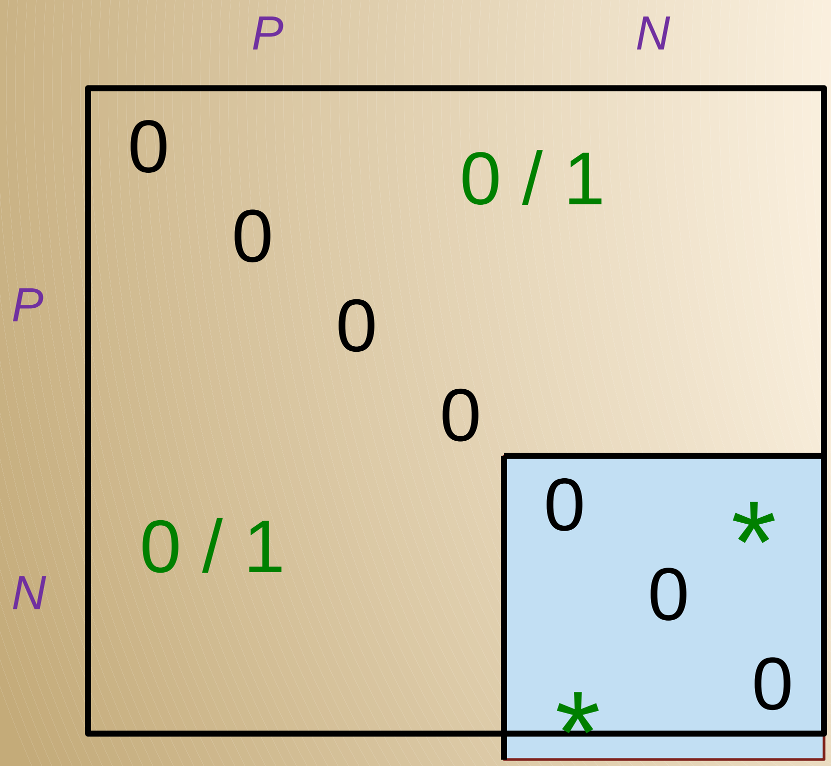
Theorem. *The rectangular block decomposition sandwich problem can be solved in O(mn) time.*

Theorem. *Square block sandwich problem is NP-complete.*

Partitioned Probe Graphs

a special case of the Sandwich Problem

All optional edges are concentrated within
one independent set *N*,
i.e., $E^0 = N \times N$.



Example: **The Probe Game**

- Take and interval graph.
- Choose a subset of vertices *N*.
- Erase the edges in $N \times N$.
- Give this Probe Problem to your students to solve. (Fill in the missing edges.)

Partitioned Probe Graphs

a special case of the Sandwich Problem

Think of *N* × *N* *as a hole in the sandwich graph that needs to be completed.*

Many Known Results on Partitioned Probe Problems

- Partitioned ***Interval Probe*** is polynomial

  Julie Johnson and Jerry Spinrad [2001], McConnell and Spinrad [2002],
  Ross McConnell and Yahav Nussbaum [2009]

- Partitioned **Chordal Probe** is polynomial

  Anne Berry, Martin Golumbic, Marina Lipshteyn [2006]

- Partitioned **Unit Interval Probe** is linear

  Yahav Nussbaum [2013]

- Two characterizations of **Chain Probe** so polynomial

  Van Bang Le [2011]

Summary

CHAIN GRAPH RECOGNITION **Linear**	**CHAIN GRAPH PARTITIONED PROBE** **Polynomial**
CHAIN GRAPH NON-PARTITIONED PROBE $O(n^2)$	**CHAIN GRAPH SANDWICH** **NP-Complete**

Other Sandwich Problems

- Hypergraph sandwich problems
- Boolean function completion problems
- **Peanut Butter** Sandwich Problems
- **Poset**

  sandwich

  problems

On Poset Sandwich Problems

Michel Habib
David Kelly
Emmanuelle Lebhar
Christophe Paul

April 2003

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Can transitive orientation make sandwich problems easier?

Michel Habib[a], David Kelly[b], Emmanuelle Lebhar[c,1], Christophe Paul[a,*,1]

The **comparability graph** *sandwich problem*:

Input: *Undirected graphs* $G^1 = (V, E^1)$, $G^2 = (V, E^2)$ s.t. $E^1 \subseteq E^2$

Question: Is there a **transitively orientable graph** $G = (V,E)$

with $E^1 \subseteq E \subseteq E^2$?

This problem is NP-Complete.

The **transitive digraph** *sandwich problem*:

Input: *Directed graphs* $D^1 = (V, F^1)$, $D^2 = (V, F^2)$ s.t. $F^1 \subseteq F^2$

Question: Is there a **transitive digraph** $D = (V,F)$

with $F^1 \subseteq F \subseteq F^2$?

This problem is EASY !

Just check that the transitive closure F^* of F is contained in F^2

So the INTERESTING **poset sandwich problems**

to be studied place a <u>*special property*</u> *on the sandwich F*:

For example,

- *interval order* sandwich problem
- *dimension 2 poset* sandwich problem
- *series-parallel poset* sandwich problem
- *semiorder* sandwich problem
- *lattice* sandwich problem (not lettuce)

Moreover, there are <u>two versions</u> of the poset sandwich problem –

depending on the INPUT: a Digraph or a Poset.

Habib, et al. [2007]:

Let Π be a **poset property**.

DIGRAPH SANDWICH PROBLEM FOR POSET PROPERTY Π

Input: Two <u>digraphs</u> $D^1 = (V, F^1)$, $D^2 = (V, F^2)$ such that $F^1 \subseteq F^2$

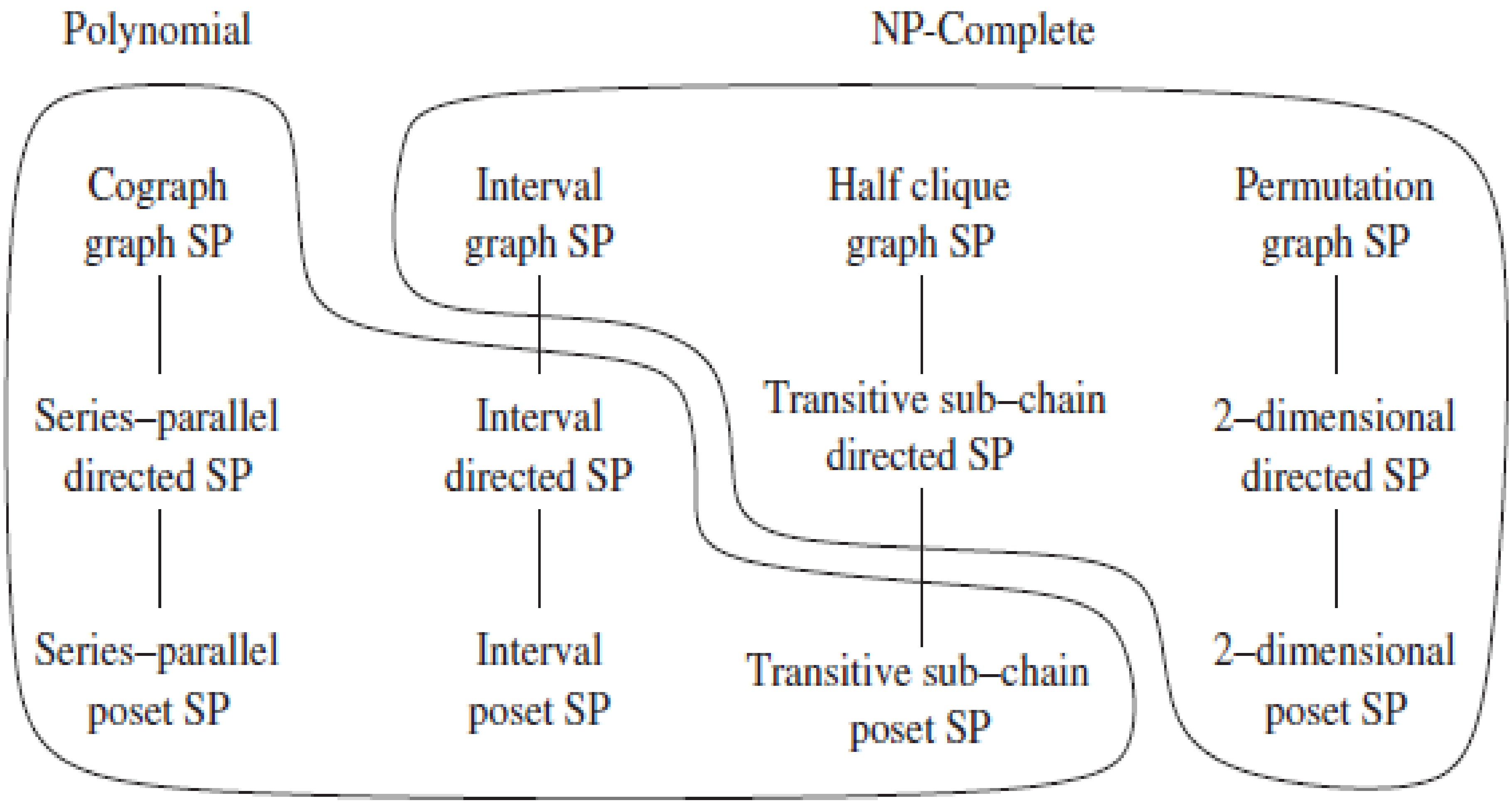
Question: Does there exist a poset $P = (V,F)$ satisfying Π with $F^1 \subseteq F \subseteq F^2$?

POSET SANDWICH PROBLEM FOR POSET PROPERTY Π

Input: Two <u>posets</u> $P^1 = (V, F^1)$, $P^2 = (V, F^2)$ such that $F^1 \subseteq F^2$

Question: Does there exist a poset $P = (V,F)$ satisfying Π with $F^1 \subseteq F \subseteq F^2$?

Habib, et al. [2007]:



Enlarging the Graph Sandwich Hierarchy

H (on 4 vertices)	Complexity of H-free Sandwich	References
P_4 (cographs)	Polynomial	GKS95
K_4 (complete)	Polynomial	easy exercise
C_4 (hole)	NP-complete	DFST11
$K_4 \backslash e$ (diamond)	Polynomial	DFST11 (see example in section 9.1)
$K_{1,3}$ (claw)	NP-complete	DFMT13
$P_3 + K_1$ (co-pan)	Polynomial	DFMT13

MORE Enlarging of the Graph Sandwich Hierarchy

Graph Class	Complexity of Sandwich	References
tolerance graphs	NP-complete	Exercise 1.4
trapezoid graphs	NP-complete	GT04
strongly chordal	NP-complete	FFKS07
chordal bipartite	NP-complete	Sri08
k-trees	NP-complete	GolWa98
k-trees for fixed k	Polynomial	GolWa98
unit interval graphs	NP-complete	GoKS94
unit interval with bounded clique size	Polynomial	KapSh96

Even MORE Enlarging of the Graph Sandwich Hierarchy

Graph Class	Complexity of Sandwich	References
caterpillars	NP-complete	ADS98,01
hereditary clique-Helly graphs	NP-complete	DPTF08
P4-sparse graphs	Polynomial	DKMM09
P4-reducible graphs	Polynomial	CKKLP05
homogeneous set	Polynomial	CEFK98
clique cutset	NP-complete	TF06
star cutset	Polynomial	TF06
skew cutset	NP-complete	DFMT13
line graphs	open	According to DFMT13

Even MORE Enlarging of the Graph Sandwich Hierarchy

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Complexity issues for the sandwich homogeneous set problem

Arnaud Durand[a], Michel Habib[b,*]

A *module* H of vertices of a graph $G = (V, E)$ is such that each vertex of $V \setminus H$ is adjacent to all vertices of H or to none of them.

A *non-trivial module* is such that $|H| \geq 2$ and $|V \setminus H| \geq 1$. A non-trivial module is also called a *homogeneous set*.

Sandwich Homogeneous Set (SHS) Problem

Input: *Undirected graphs* $G^1 = (V, E^1)$, $G^2 = (V, E^2)$ s.t. $E^1 \subseteq E^2$

Question: Is there a **sandwich graph** $G = (V,E)$

for the pair (G^1, G^2) containing a homogeneous set H ?

Homogeneous Sandwich Problem

$O(n^4)$ Cerioli, Everett, de Figuereido, Klein (1998)

$O(n^3 \log n)$ de Figueiredo, Fonseca, de Sa, Spinrad (2006)

Durand and Habib (2011):

this implies that finding the maximum sandwich homogeneous set is NP-hard.

the counting version of this problem, which is proved to be #P-complete.

MAX Sandwich Homogeneous Set (SHS) Problem

Input: *Undirected graphs* $G^1 = (V, E^1)$, $G^2 = (V, E^2)$ s.t. $E^1 \subseteq E^2$

and an integer k.

Question: Does there exist a **sandwich homogeneous set** ***H***

such that |H| >= k ?

Theorem. Max sandwich homogeneous set problem is NP-complete.

Using Hastad's result on Max-independent set, and a tight quasi-linear reduction from the k-independent set problem to the sandwich homogeneous set of size k,
they also obtain:

Theorem. Max sandwich homogeneous set problem cannot be approximated within a factor of $n^{1-\epsilon}$ for any $\epsilon > 0$

seconds on the history of FRENCH graph theory 60

FASCICULE XVIII

Les Réseaux (ou graphes)

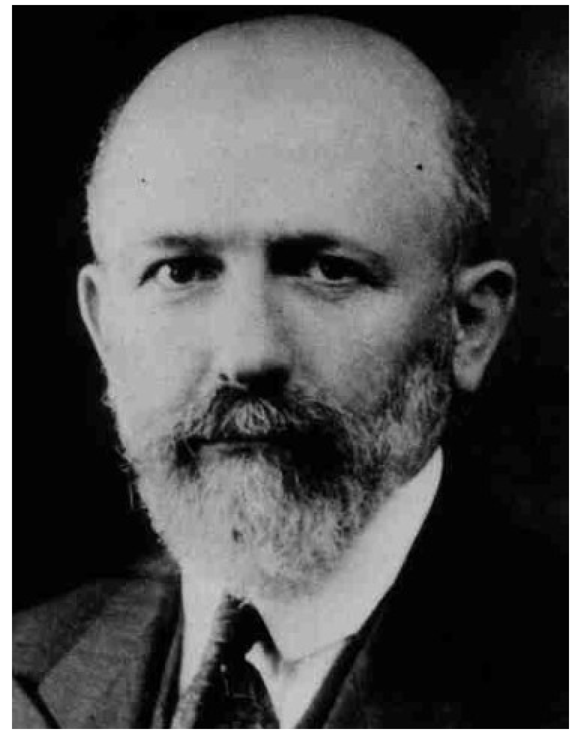
PAR M. A. SAINTE-LAGUË

Professeur au Lycée Carnot.

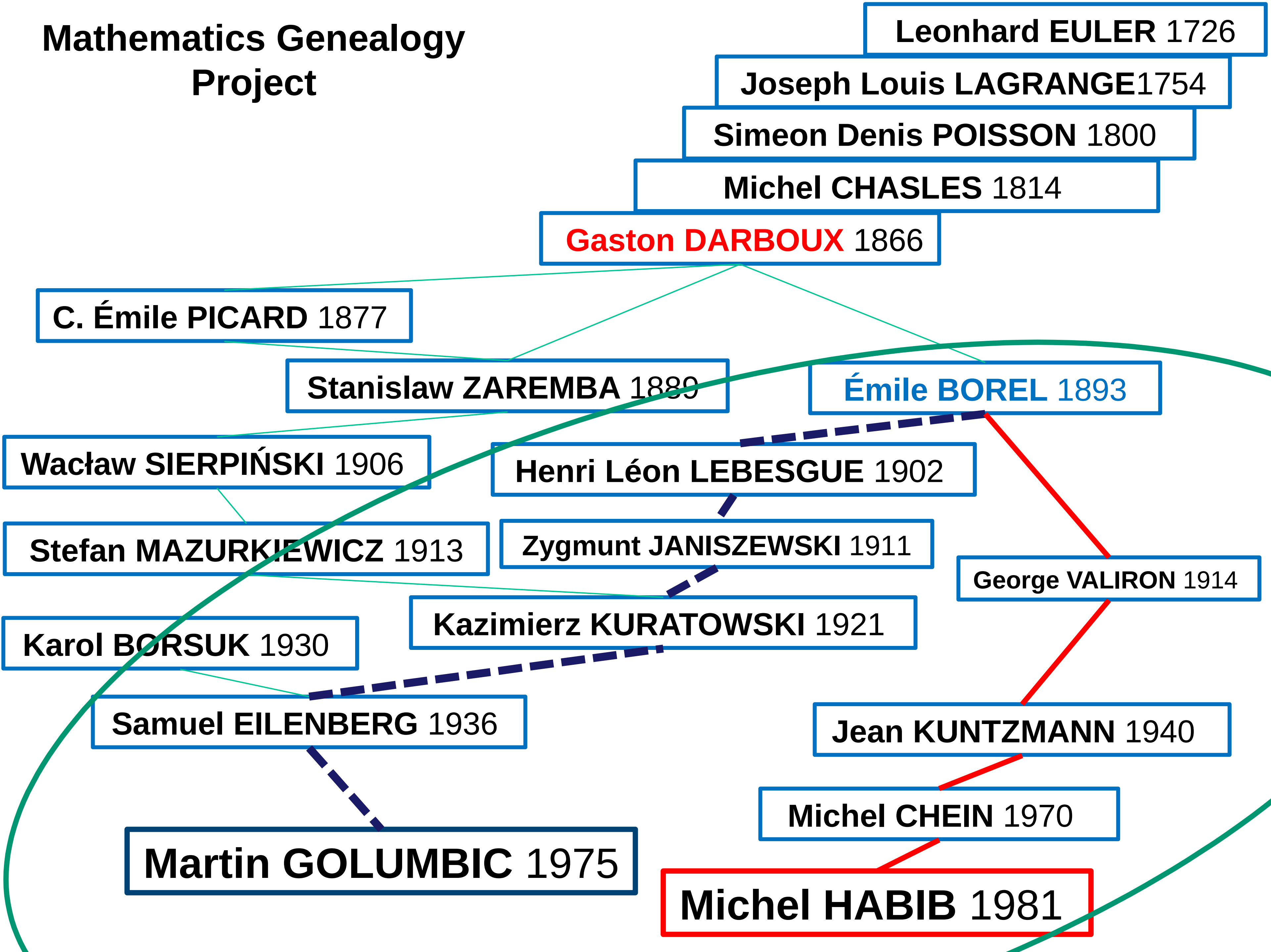
PARIS

GAUTHIER-VILLARS ET C^ie, ÉDITEURS
LIBRAIRES DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE
Quai des Grands-Augustins, 55.
1926

The Zeroth Book of Graph Theor



Mathematics Genealogy Project



Thank you MICHEL