Interval-like Graphs and Digraphs

Pavol Hell, Simon Fraser University

Michel-fest (40 années d'algorithmique de graphes)

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Co-authors

Results joint with

- Arash Rafiey
- Tomás Feder
- Jing Huang
- Ross McConnell

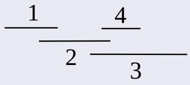
Interval Graphs

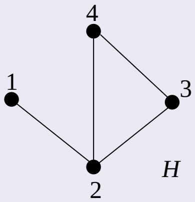
Interval graph

Vertices v can be represented by intervals I_v, so that

$$v \sim w \iff I_v \cap I_w \neq \emptyset$$

Example





Interval Graphs

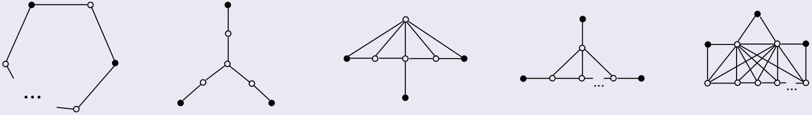
Appeal

- Pervasive applications Benzer 1959, ..., Klee 1969
- Efficient recognition algorithms Booth-Lueker 1976, ..., Corneil-Olariu-Stewart 1998
- Efficient optimization algorithms Gavril 1974, Rose-Tarjan-Lueker 1976, ...
- Elegant characterizations Lekkerkerker-Boland 1962, ...

Structural characterization

Lekkerkerker-Boland 1962

H is an interval graph $\iff$ H has no induced subgraph from



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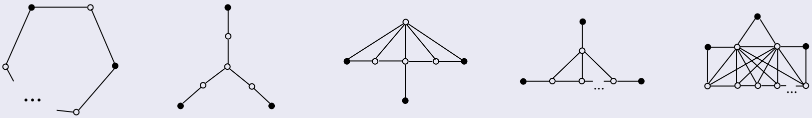

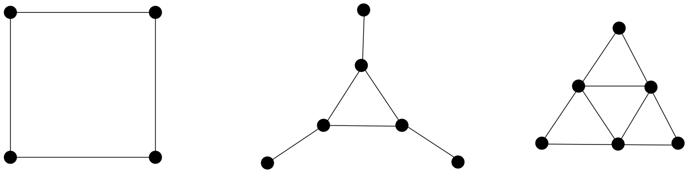

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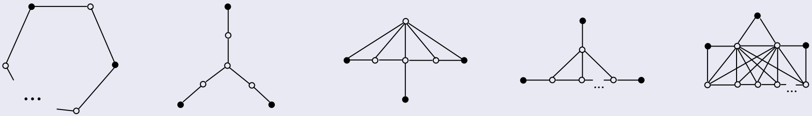
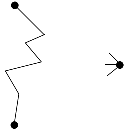


Lekkerkerker-Boland 1962

H is an interval graph $\iff$ H has no AT or induced C_4, C_5.



An Ordering Characterization

Theorem

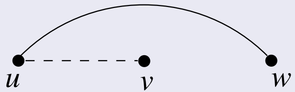
H is an interval graph

$$\iff$$

$V(H)$ can be linearly ordered by $<$ so that

$$u \sim w \text{ and } u < v < w \implies u \sim v$$

Dotted edge cannot be absent



An Ordering Characterization

Min-ordering

H is an interval graph

$$\iff$$

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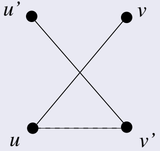


An Ordering Characterization

Min-ordering

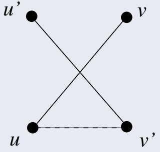
H is an interval graph

$$\Longleftrightarrow$$

$V(H)$ can be linearly ordered by $<$ so that

$$u \sim v, u' \sim v' \implies \min(u, u') \sim \min(v, v')$$

Dotted edge cannot be absent



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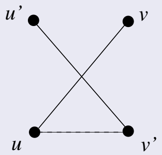
Proof of $\implies$: order by left endpoints



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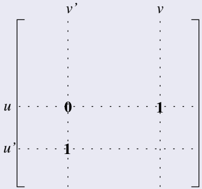
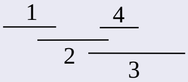
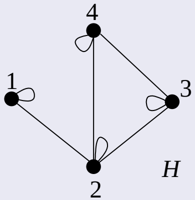
Augmented adjacency matrix doesn't contain a bad 0



Observation

Interval graphs are reflexive (have all loops)





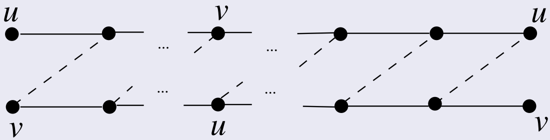
Characterization Theorems

The following statements are equivalent for a reflexive graph

1. H is an interval graph
2. H has a min ordering
3. Adjacency matrix doesn't contain a bad 0

An Obstruction to Min Ordering

Invertible pair



Dashed line = non-edge

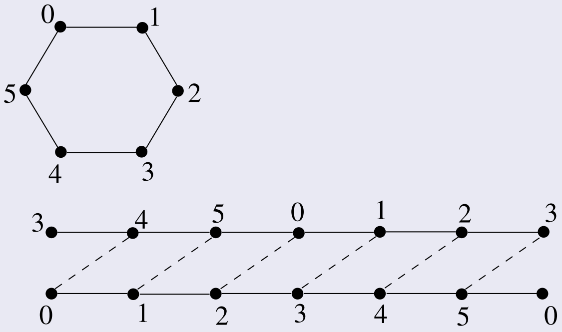
Characterization Theorems

The following statements are equivalent for a reflexive graph

1. H is an interval graph
2. H has a min ordering
3. Adjacency matrix doesn't contain a bad 0
4. H has no invertible pair
5. H has no AT or induced C_3, C_4

An Obstruction to Min Ordering

A graph with an invertible pair cannot have a min ordering



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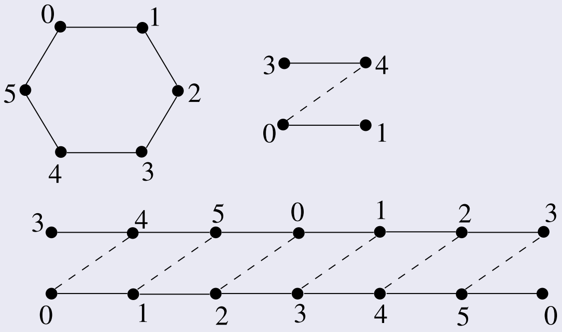

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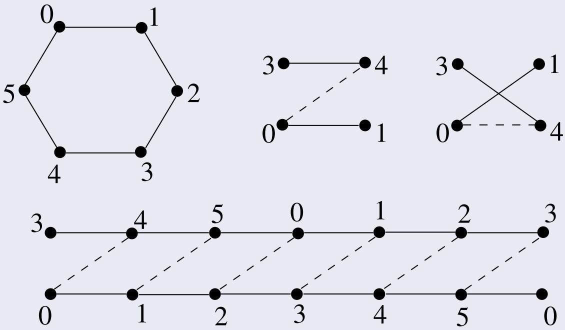
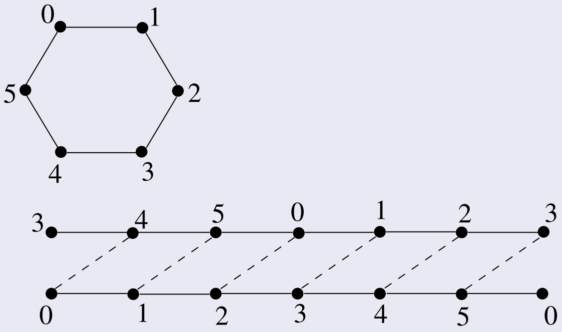


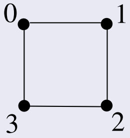
An Obstruction to Min Ordering

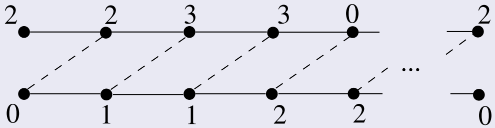
$0, 3$ is an invertible pair in C_6



An Obstruction to Min Ordering

Looking for an invertible pair in C_4





Digraphs

Digraphs

An interval digraph

Vertices can be represented by pairs of intervals I_v, J_v, so that

$$v \to w \iff I_v \cap J_w \neq \emptyset$$

Example

Sen-Das-Roy-West 1989

Digraphs

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Example

Sen-Das-Roy-West 1989

No obstruction characterization; $O(n^{15})$ recognition Mueller 1997

Faster algorithms claimed by Rafiey 2013 and by Das 2013

A min ordering of H

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Digraphs

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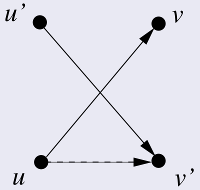

Reflexive Digraphs

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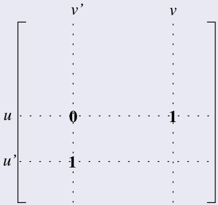

Reflexive Digraphs

Reflexive Digraphs

A geometric representation

A reflexive digraph has has a min ordering $\iff$ it is an adjusted-interval digraph

Feder+H+Huang+Rafiey 2012

Reflexive Digraphs

A geometric representation

A reflexive digraph has has a min ordering $\iff$ it is an adjusted-interval digraph

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Adjusted-interval digraphs

Vertices can be represented by pairs of *adjusted* intervals I_v, J_v,

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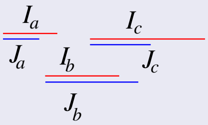
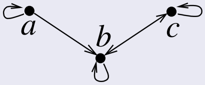


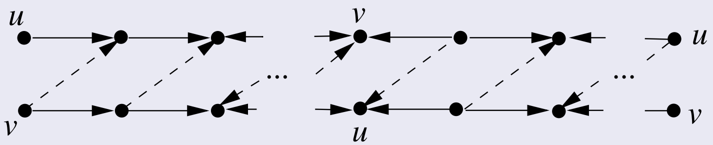


A reflexive digraph H an adjusted-interval digraph if and only if

Obstruction Characterization

A reflexive digraph H an adjusted-interval digraph if and only if

it has no invertible pair



Adjusted-interval Digraphs

Similarities to interval graphs

- similar geometric representations
- similar obstructions
- similar ordering characterization

Adjusted-interval Digraphs

Similarities to interval graphs

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- similar obstructions
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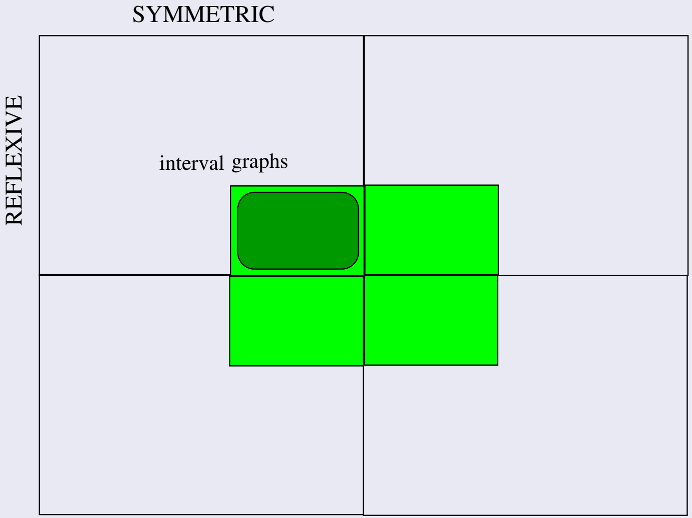
$O(n^4)$ recognition algorithm

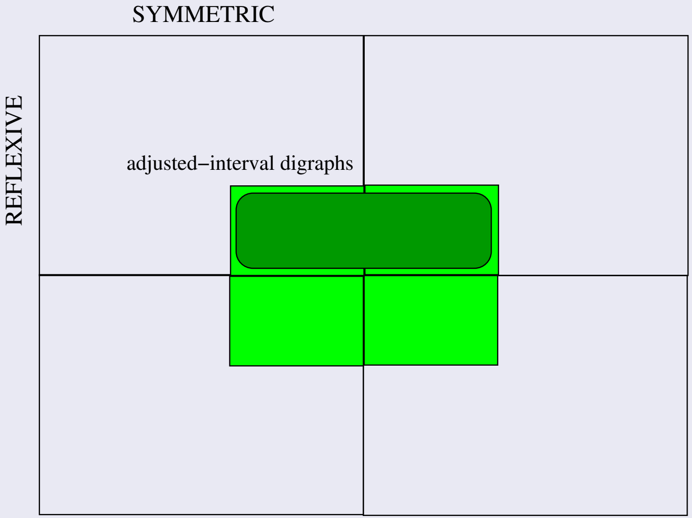
Open

A more efficient recognition algorithm?

Interval-like digraphs



Interval-like digraphs



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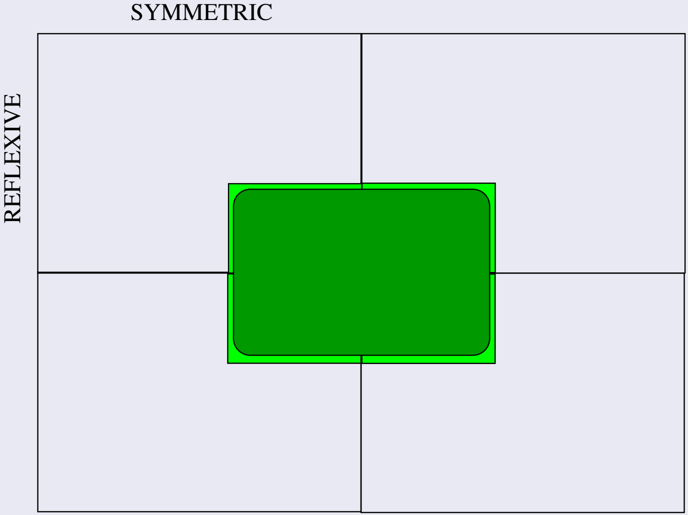

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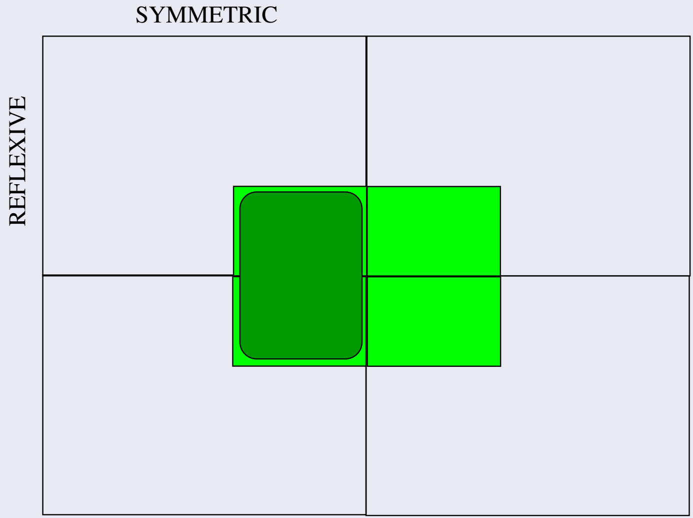

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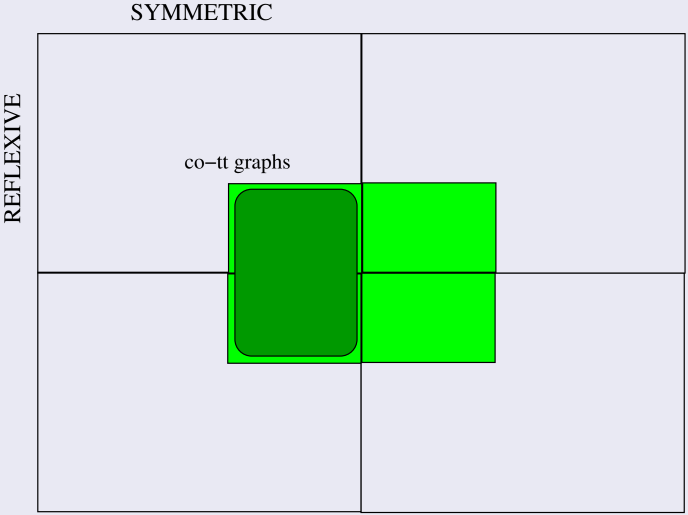

Co-tt Graphs

Threshold tolerance (tt-) graphs

Each vertex v can be assigned a *weight* w_v and a *threshold* t_v so that

$$u \sim v \iff w_u + w_v \geq \min(t_u, t_v)$$

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Co-tt graphs = complements of tt-graphs

H is a co-tt graph $\iff$ there exist real functions ℓ, r on $V(H)$ such that

$$u \sim v \iff \ell(u) \leq r(v) \text{ and } \ell(v) \leq r(u)$$

Monma+Reed+Trotter 1988

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Positive (blue) and negative (red)

- Positive intervals (blue vertices) have $\ell(v) \leq r(v)$
- Negative intervals (red vertices) have $\ell(v) > r(v)$

Co-tt Graphs

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Positive (blue) and negative (red)

- Positive intervals (blue vertices) have $\ell(v) \leq r(v)$
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Corollary

A co-tt graph has loops at all blue vertices (an none at red vertices).

Co-tt Graphs

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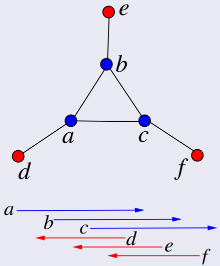
Adjacency rules

- Blue u and blue v have $u \sim v \iff$ their intervals intersect
- Red u and blue v have $u \sim v \iff$ u's interval is contained in v's interval

Co-tt Graphs

Example



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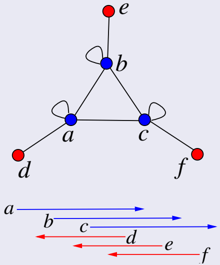

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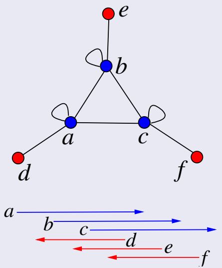
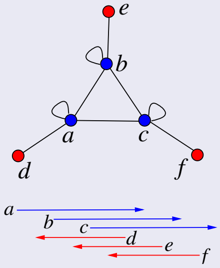


A co-tt model

- Blue = interval graph
- Red = independent set

Co-tt Graphs

Example

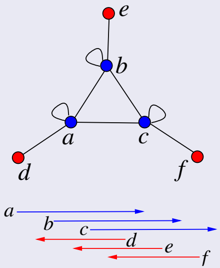


A symmetric digraph H is a co-tt graph $\iff$

Vertices can be assigned intervals, positive for vertices with loops and negative for vertices without loops, so that the adjacency rules hold

Co-tt Graphs

Example



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Co-tt Graphs

Golumbic+Weingarten+Limouzy 2014

If a graph H is a co-tt graph under a red-blue colouring, then it is a co-tt graph under a 'special' colouring – where all non-twin simplicial vertices are red, and all other vertices are blue

Co-tt Graphs

Golumbic+Weingarten+Limouzy 2014

If a graph H is a co-tt graph under a red-blue colouring, then it is a co-tt graph under a 'special' colouring – where all non-twin simplicial vertices are red, and all other vertices are blue

Equivalently

If loops can be added to a graph H to create a co-tt graph, then they can be added to just the vertices that are not simplicial, and those that are simplicial but have twins.

Co-tt Graphs

A symmetric digraph H is a co-tt graph $\iff$

- H admits a min ordering $\iff$
- Adjacency matrix contains no bad 0 $\iff$
- H admits an invertible pair

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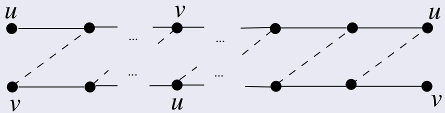
Invertible pair



H+Huang+McConnell 2018

Min Ordering

Special graphs with min ordering

- A symmetric digraph has a min ordering if and only if it is a co-tt graph

Min Ordering

Special graphs with min ordering

- A symmetric digraph has a min ordering if and only if it is a co-tt graph
- A reflexive digraph has a min ordering if and only if it is an adjusted-interval digraph

Min Ordering

Special graphs with min ordering

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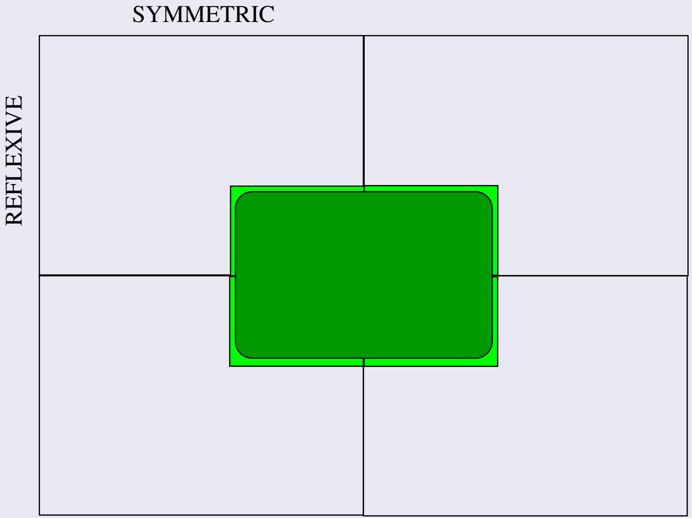
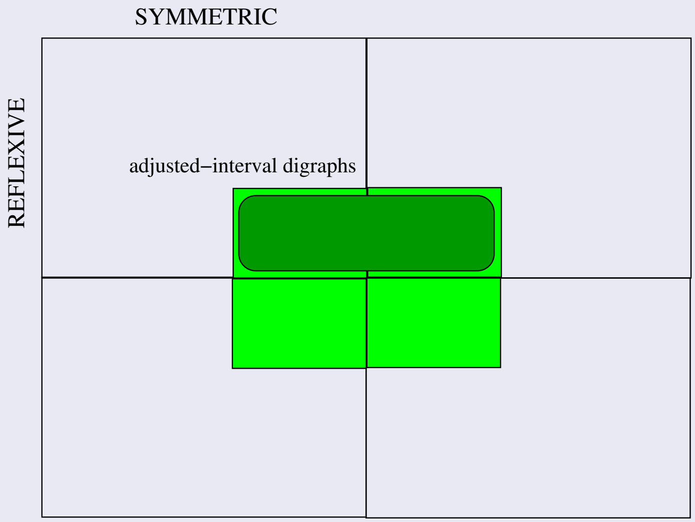
- A reflexive symmetric digraph has a min ordering if and only if it is an interval graph

Interval-like digraphs



The World of Digraphs

Interval-like digraphs



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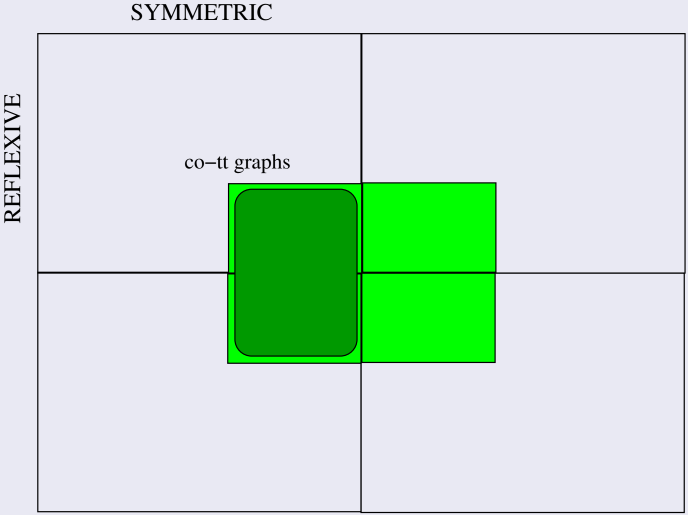

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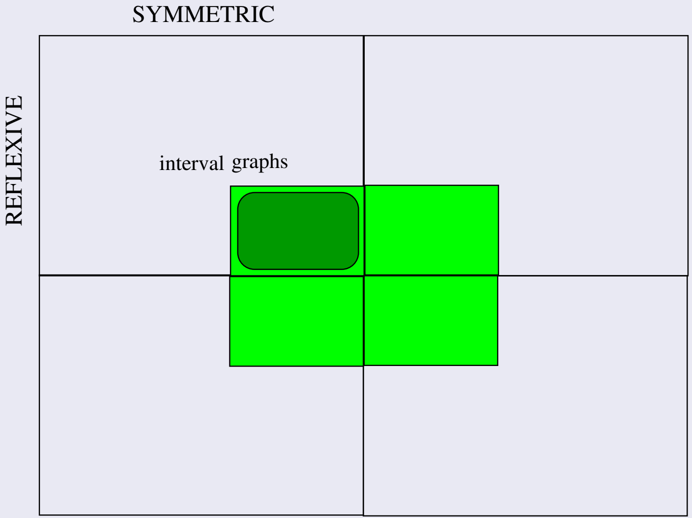

Interval-like digraphs

Min-orderable digraphs

Interval-like digraphs

Min-orderable digraphs

- Geometric representation?
- Obstruction characterization?
- Polynomial recognition algorithm?

Min Orderable Digraphs

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Min Orderable Digraphs

Adjusted-interval digraphs

H is an adjusted-interval digraph $\iff$ there exist real functions ℓ, r, s on $V(H)$ such that each $\ell(v) \leq r(v), \ell(v) \leq s(v)$ and

$$u \to v \iff \ell(u) \leq s(v) \text{ and } \ell(v) \leq r(u)$$

Min Orderable Digraphs

Adjusted-interval digraphs

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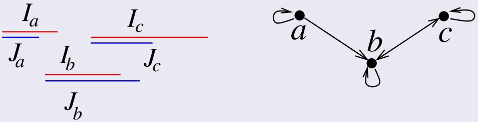
Signed-interval digraphs

H is a *signed-interval digraph* if there exist real functions ℓ, r, s on $V(H)$ such that

$$u \to v \iff \ell(u) \leq s(v) \text{ and } \ell(v) \leq r(u)$$

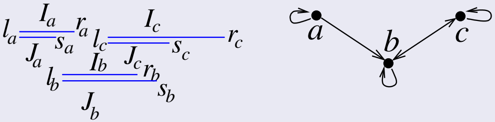
Signed-interval digraphs

Adjusted-interval digraph



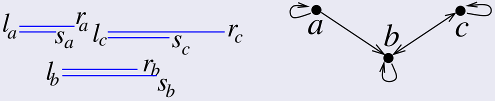
Signed-interval digraphs

Adjusted-interval digraph



Signed-interval digraphs

Adjusted-interval digraph



Signed-interval digraphs

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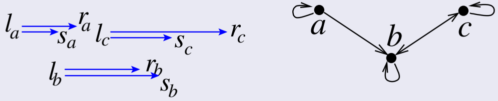

Signed-interval digraphs

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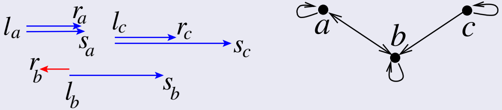

General Digraphs

H is a signed-interval digraph $\iff$

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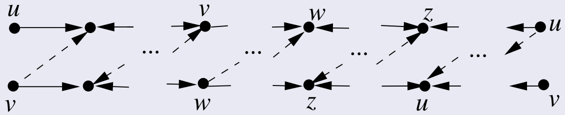

H+Huang+McConnell 2018, H+Rafiey 2018+

General Digraphs

H is a signed-interval digraph $\iff$

- H admits a min ordering $\iff$
- Adjacency matrix contains no bad 0 $\iff$
- H has no invertible circuit
- H is a bi-arc digraph

A bi-arc digraph H

Representable by two consistent families of circular arcs

$$I_v, v \in V(H), \text{ and } J_v, v \in V(H),$$

$$uv \in E(H) \iff I_u \cap J_v = \emptyset$$

A bi-arc digraph H

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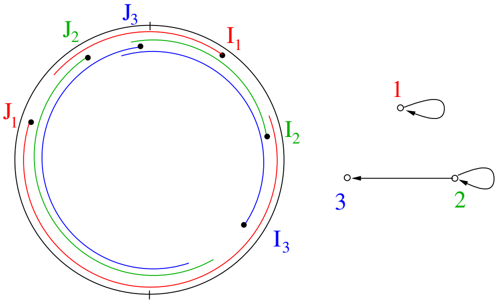
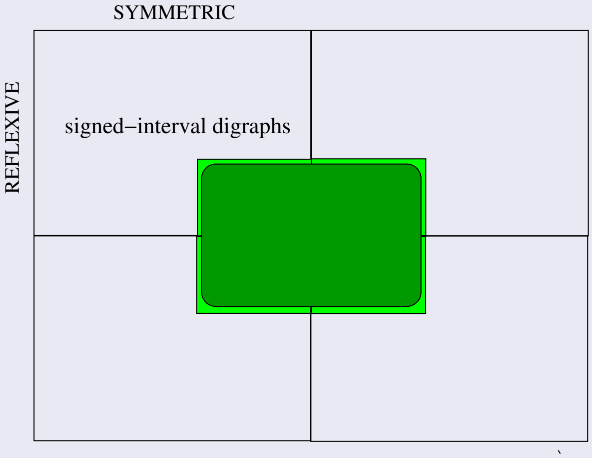


Signed-interval digraphs

Signed-interval digraph

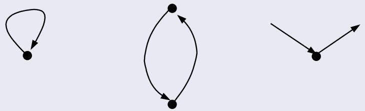


Signed-interval digraphs

	SYMMETRIC		ANTISYMMETRIC
REFLEXIVE			
IRREFLEXIVE			

Bipartite digraphs

Forbidden situations



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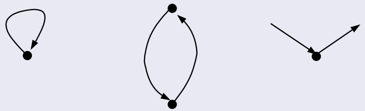
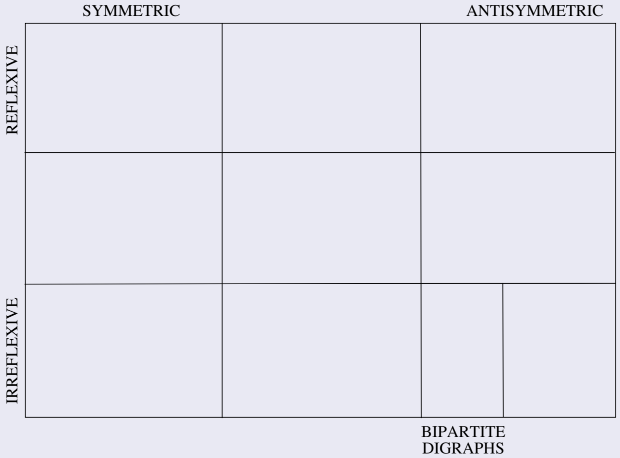


A bipartite digraph

Has only sources and sinks

Signed-interval digraphs

Signed-interval digraphs



Signed-interval digraphs

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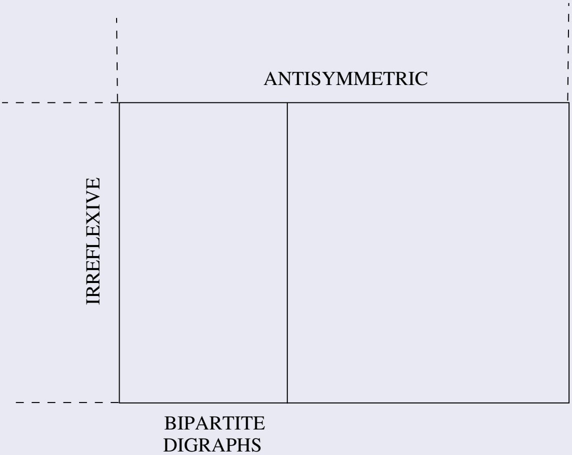

Signed-interval digraphs

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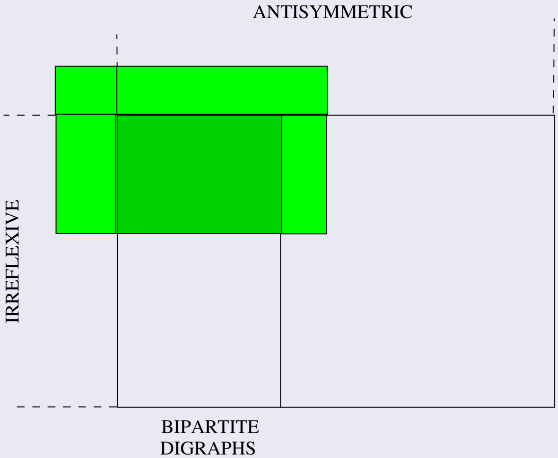

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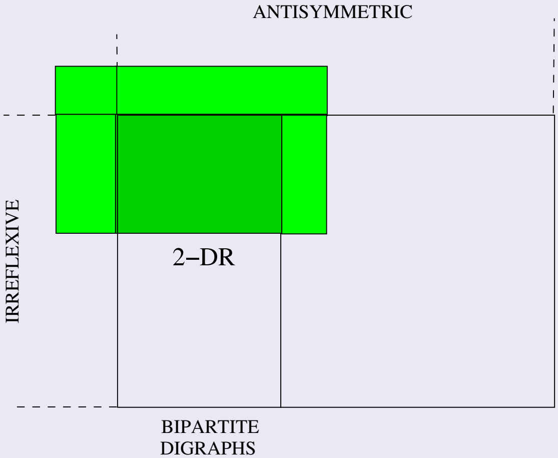
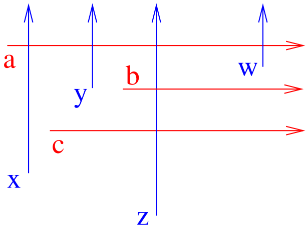
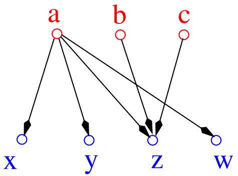


Two Directional (Orthogonal) Ray Graphs

A 2-DR graph

Intersection graph of a family of UP and RIGHT rays





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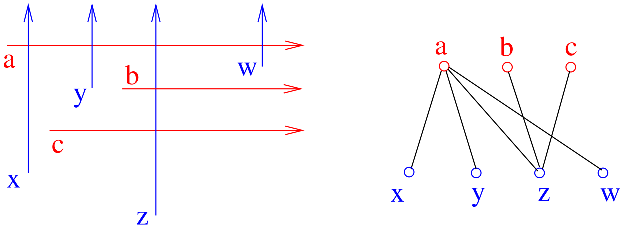

New Fact

Special signed-interval digraphs

- A symmetric digraph is a signed-interval digraph if and only if it is a co-tt graph
- A reflexive digraph is a signed-interval digraph if and only if it is an adjusted-interval digraph
- A reflexive symmetric digraph is a signed-interval digraph if and only if it is an interval graph

New Fact

Special signed-interval digraphs

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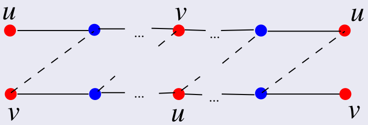
- A reflexive digraph is a signed-interval digraph if and only if it is an adjusted-interval digraph
- A reflexive symmetric digraph is a signed-interval digraph if and only if it is an interval graph
- A bipartite digraph is a signed-interval digraph if and only if if it is a 2-DR graph

Two Directional (Orthogonal) Ray Graphs

A bipartite digraph H is a 2DR graph if and only if

it has no invertible pair



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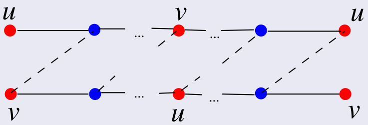

A bipartite digraph H is a 2DR graph if and only if

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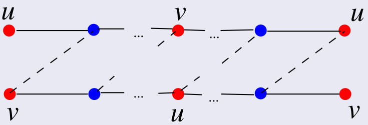


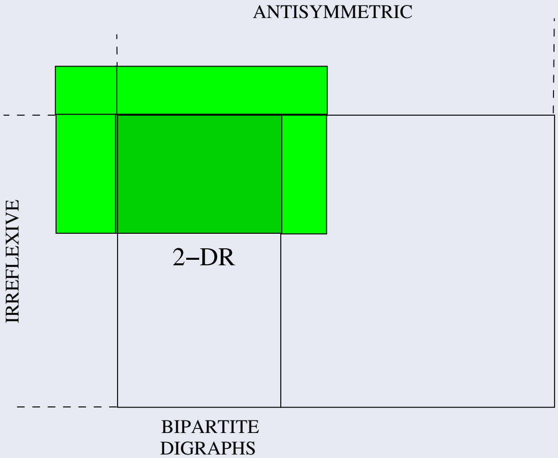
A bipartite digraph H is a 2DR graph if and only if

- H is an interval containment bigraph
- $\overline{H}$ is a circular arc graph

Feder, H and Huang 1999, Huang 2006, Shrestha, Tayu, and Ueno 2010, H+Rafiey 2011

Signed-interval digraphs

Signed-interval digraphs



Two Directional (Orthogonal) Ray Graphs

A better analogue of interval graphs than interval bigraphs

Ordering characterization, obstruction characterization, more efficient recognition

Two Directional (Orthogonal) Ray Graphs

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More general than interval bigraphs

- H is a 2DR graph $\iff$ $\overline{H}$ is a circular arc graph
- H is an interval bigraph $\iff$ $\overline{H}$ is a circular arc graph that can be represented without two arcs covering the circle

H and Huang 2004

Thank you

Happy Retirement, Michel