Vertex Orderings and Decomposition into Cographs

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— 40 Years of Graphs and Algorithms —
Orderings
Local Search Ordering

BFS

LexBFS

DFS

LexBFS

Sorry Michel...not going to look at these!
Local Search Ordering

BFS

LexBFS

DFS

LexBFS

😊 Sorry Michel...not going to look at these!
Non local ordering with local properties
Generalized Coloring Numbers

\[ \text{adm}_r(G) \leq \text{col}_r(G) \leq \text{wcol}_r(G) \leq 1 + r(\text{adm}_r(G) - 1)r^2 \]
### Bounds

<table>
<thead>
<tr>
<th>Class of graphs</th>
<th>$\text{col}_r$</th>
<th>$\text{wcol}_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bounded expansion</td>
<td>$\leq f(r)$ (Zhu ’09)</td>
<td></td>
</tr>
<tr>
<td>No $K_t$-minor</td>
<td>$\binom{t-1}{2}(2r + 1)$</td>
<td>$\binom{r+t-2}{t-2}(t - 3)(2r + 1)$</td>
</tr>
<tr>
<td>Planar</td>
<td>$5r + 1$</td>
<td>$(r+2)_2(2r + 1)$</td>
</tr>
</tbody>
</table>

(van den Leuven, POM, Quiroz, Rabinovich, Siebertz ’17)
Powers of Sparse Graphs
Subcoloring

Theorem (Nešetřil, POM, Zhu ’18+)

For every graph $G$ and every integer $k \geq 2$ we have

$$\chi_{sub}(G^k) \leq \begin{cases} 
\text{wcol}_{2k-1}(G) & \text{if } k \text{ is odd}, \\
\text{wcol}_{2k}(G) & \text{if } k \text{ is even}.
\end{cases}$$

Corollary

Let $k \geq 2$ and $\mathcal{D} = \{H \subseteq_i G^k \mid G \in \mathcal{C}\}$. Tfae:

1. $\mathcal{C}$ has bounded expansion;
2. $\mathcal{D}$ has bounded subchromatic number;
3. $\mathcal{D}$ is linearly $\chi$-bounded;
4. $\mathcal{D}$ is $\chi$-bounded.
Proof

Let $k' = \lfloor k/2 \rfloor$ and
\[
\begin{cases}
(c, <) \text{ a rank k}+2k' \text{ weak colouring}; \\
v \mapsto \hat{v} := \min \text{ Ball}_{k'}(v); \\
\gamma(v) := c(\hat{v}).
\end{cases}
\]

\[
\begin{cases}
uv \in E(G^k) \\
\gamma(u) = \gamma(v)
\end{cases} \Rightarrow \hat{u} = \hat{v} \quad \leadsto \quad \text{No } \gamma\text{-monochromatic induced } P_3.
\]
Weak Coloring Numbers

Theorem (Nešetřil, POM, Zhu ’18+)

For every integers $k \geq 2$ and $r \geq 1$, every graph $G$, and every induced subgraph $H$ of $G^k$ we have

$$wcol_r(H) \leq wcol_{kr+2\lfloor \frac{k}{2} \rfloor}(G) \omega(H).$$

Corollary

$$\omega(G^k) \leq \chi(G^k) \leq \text{col}(G^k) \leq wcol_{2k}(G) \omega(G^k).$$

Problem

PTAS for $\omega(G^k)$?
Structural Sparsity
Structurally sparse graphs

A class is *structurally sparse* if it can be (simply) interpreted in a sparse class.

\[ l(G) \models x \sim y \iff G \models \exists z_1, z_2 \left( (x \sim z_1) \wedge (z_1 \sim z_2) \wedge (z_2 \sim y) \right) \]
Motivation: Model Checking

Theorem (Gajarský, Hliněný, Lokshtanov, Ramanujan ’16)
Let $\mathcal{D}$ be a graph class interpretable in a bounded degree class. Then $\mathcal{D}$ has an FO model checking algorithm in FPT.

Conjecture (Gajarský et al. 2016)
Let $\mathcal{C}$ be a nowhere dense class and $\mathcal{D}$ a graph class interpretable in $\mathcal{C}$. Then $\mathcal{D}$ has an FO model checking algorithm in FPT.
### Structural Sparsity

#### Orderings

- Powers
- Structural Sparsity
- Sparsification
- Next Step?

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**Monotone**

- Simple First-Order Interpretation
- Bounded Expansion
  - Tree-depth
  - Tree-depth decompositions
  - \( \chi \)
  - Shallow topological minor

**Hereditary**

- Structurally Bounded Expansion
  - SC-depth
  - SC-depth decompositions
  - \( \chi/\omega \)
  - Shallow vertex minor

- NIP
- Stability
- VC-density

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Theorem (Gajarský, Kreutzer, Kwon, Nešetril, POM, Pilipczuk, Siebertz, Toruńczyk 2018)

A class of graphs has **SC-depth decompositions** if and only if it has a **structurally bounded expansion**.
Sparsification
Sparsification: Cographs strike back!
Vertex bloc: bounded depth cographs
Edge bloc: bounded depth bi-cographs
(c, d)-fold coloring
(c, d)-fold coloring
Sparsification: Cut & Paste
Theorem (Gajarský, Kreutzer, Kwon, Nešetril, POM, Pilipczuk, Siebertz, Toruńczyk 2018)

For a class of graphs $\mathcal{C}$ with $(c, d)$-fold coloring the following are equivalent:

- $\mathcal{C}$ has SC-depth decompositions
- $\text{Sparsify}(\mathcal{C})$ has tree-depth decompositions;
- $\text{Sparsify}(\mathcal{C})$ has bounded expansion.
- $\mathcal{C}$ has structurally bounded expansion;

If $(c, d)$-fold colorings can be computed in time $F(n)$ for $G \in \mathcal{C}$ then checking a first-order sentence $\phi$ on $\mathcal{C}$ can be done in time

$$F(n) + C(\phi, \mathcal{C})n.$$
Next Step?
Problem

Let $C$ be a **structurally bounded expansion** class.

Is it possible to compute an $(c, f)$-fold coloring for graphs in $C$ in **polynomial time**?

▶ Would have quite a few algorithmic consequences!
Thank you for your attention.