Vertex Orderings and Decomposition into Cographs

Patrice Ossona de Mendez

Charles University
Praha, Czech Republic

LIA STRUCO

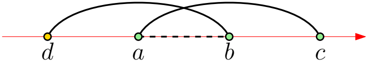
CAMS, CNRS/EHESS
Paris, France

— 40 Years of Graphs and Algorithms —

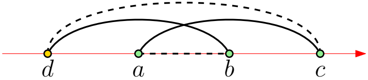
Orderings

Local Search Ordering

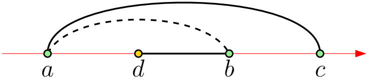
BFS



LexBFS



DFS



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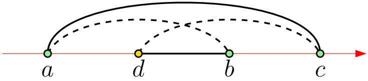
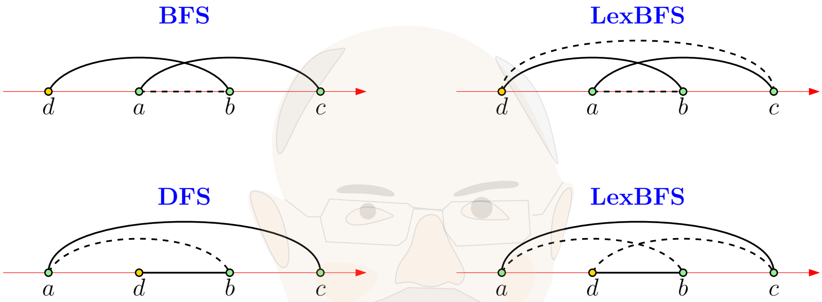
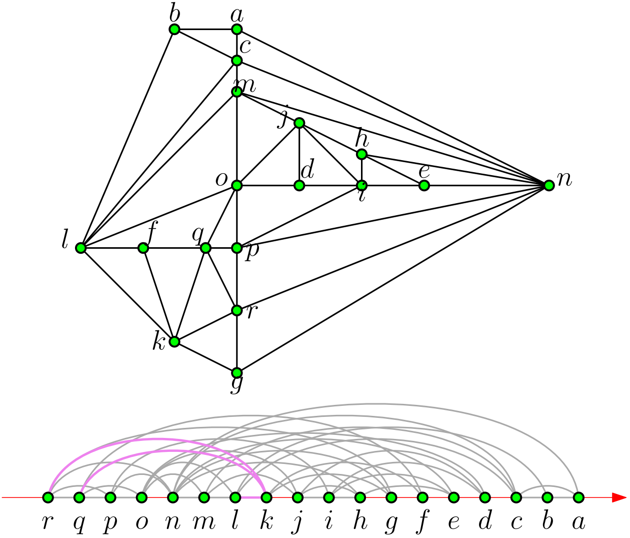


Local Search Ordering



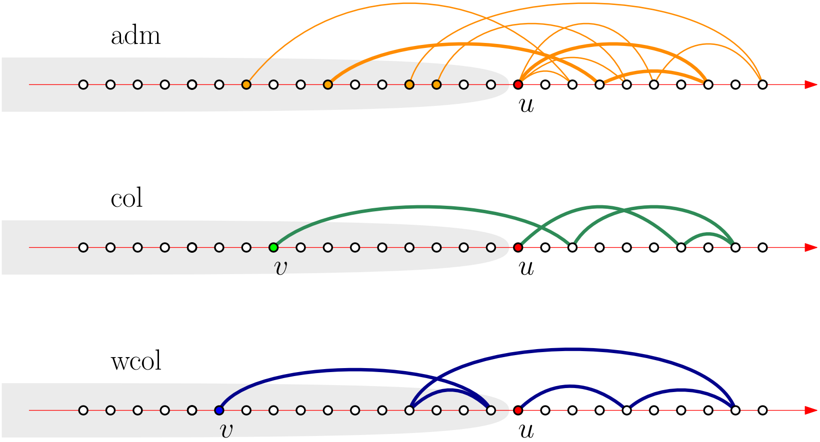
Sorry Michel... not going to look at these!

Non local ordering with local properties



Generalized Coloring Numbers

$$\mathrm{adm}_r(G) \leq \mathrm{col}_r(G) \leq \mathrm{wcol}_r(G) \leq 1 + r(\mathrm{adm}_r(G) - 1)^{r^2}$$



Bounds

Class of graphs	col_r	wcol_r
Bounded expansion	$\leq f(r)$ (Zhu '09)	
No K_t-minor	$\binom{t-1}{2}(2r+1)$	$\binom{r+t-2}{t-2}(t-3)(2r+1)$
Planar	$5r+1$	$\binom{r+2}{2}(2r+1)$

(van den Leuven, POM, Quiroz, Rabinovich, Siebertz '17)

Powers of Sparse Graphs

Subcoloring

Theorem (Nešetřil, POM, Zhu '18+)

For every graph G and every integer $k \geq 2$ we have

$$\chi_{\text{sub}}(G^k) \leq \begin{cases} \text{wcol}_{2k-1}(G) & \text{if } k \text{ is odd,} \\ \text{wcol}_{2k}(G) & \text{if } k \text{ is even.} \end{cases}$$

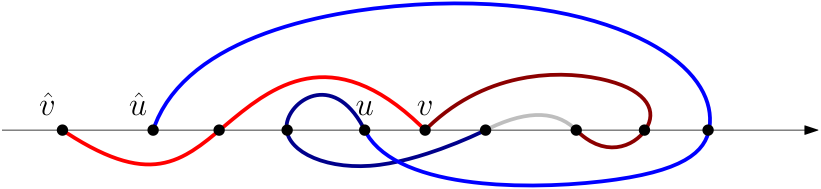
Corollary

Let $k \geq 2$ and $\mathcal{D} = \{H \subseteq_i G^k \mid G \in \mathcal{C}\}$. Tfae:

1. $\mathcal{C}$ has bounded expansion;
2. $\mathcal{D}$ has bounded subchromatic number;
3. $\mathcal{D}$ is linearly χ-bounded;
4. $\mathcal{D}$ is χ-bounded.

Proof

Let $k' = \lfloor k/2 \rfloor$ and $\begin{cases} (c,<) \text{ a rank k+2k' weak colouring;} \\ v \mapsto \hat{v} := \min \mathrm{Ball}_{k'}(v); \\ \gamma(v) := c(\hat{v}). \end{cases}$



$\begin{cases} uv \in E(G^k) \\ \gamma(u) = \gamma(v) \end{cases} \Rightarrow \hat{u} = \hat{v} \quad \rightsquigarrow$ No γ-monochromatic induced P_3.

Weak Coloring Numbers

Theorem (Nešetřil, POM, Zhu '18+)

For every integers $k \geq 2$ and $r \geq 1$, every graph G, and every induced subgraph H of G^k we have

$$\mathrm{wcol}_r(H) \leq \mathrm{wcol}_{kr+2\lfloor \frac{k}{2} \rfloor}(G)\, \omega(H).$$

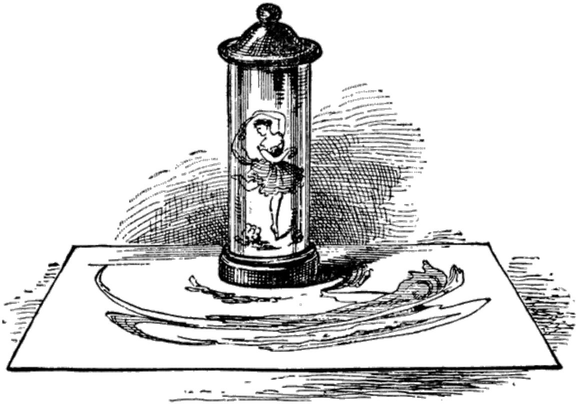
Corollary

$$\omega(G^k) \leq \chi(G^k) \leq \mathrm{col}(G^k) \leq \mathrm{wcol}_{2k}(G)\, \omega(G^k).$$

Problem

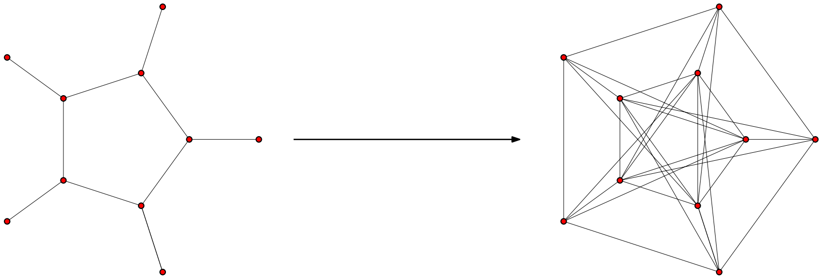
PTAS for $\omega(G^k)$?

Structural Sparsity



Structurally sparse graphs

A class is *structurally sparse* if it can be (simply) interpreted in a sparse class.



$$\mathsf{I}(G) \models x \sim y \quad \iff \quad G \models \exists z_1, z_2 \big((x \sim z_1) \wedge (z_1 \sim z_2) \wedge (z_2 \sim y)\big)$$

Motivation: Model Checking

Theorem (Gajarský, Hliněný, Lokshtanov, Ramanujan '16)

Let $\mathscr{D}$ be a graph class interpretable in a bounded degree class. Then $\mathscr{D}$ has an FO model checking algorithm in FPT.

Conjecture (Gajarský *et al.* 2016)

Let $\mathscr{C}$ be a nowhere dense class and $\mathscr{D}$ a graph class interpretable in $\mathscr{C}$. Then $\mathscr{D}$ has an FO model checking algorithm in FPT.

Structural Sparsity

Monotone | Hereditary

Simple First-Oder Interpretation

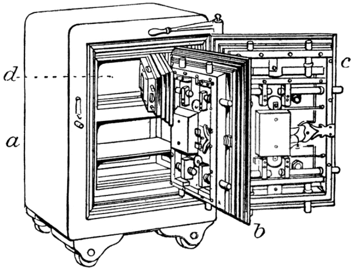
Bounded Expansion	Structurally Bounded Expansion
Tree-depth Tree-depth decompositions χ Shallow topological minor	SC-depth SC-depth decompositions χ/ω Shallow vertex minor

NIP

Stability

VC-density

Structural Sparsity

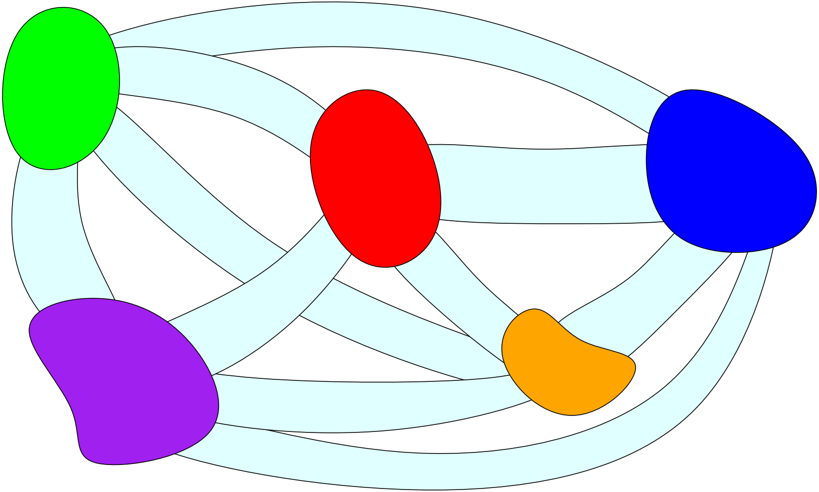


Theorem (Gajarský, Kreutzer, Kwon, Nešetril, POM, Pilipczuk, Siebertz, Toruńczyk 2018)

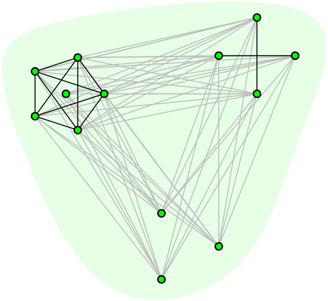
A class of graphs has SC-depth decompositions if and only if it has a structurally bounded expansion.

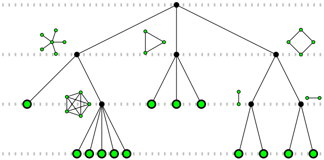
Sparsification

Sparsification: Cographs strike back!

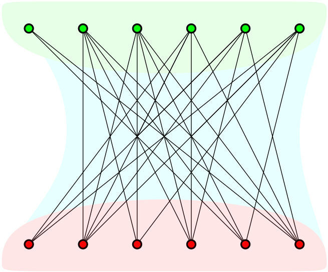


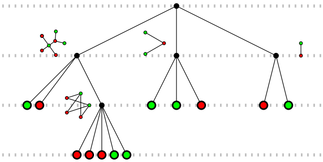
Vertex bloc: bounded depth cographs



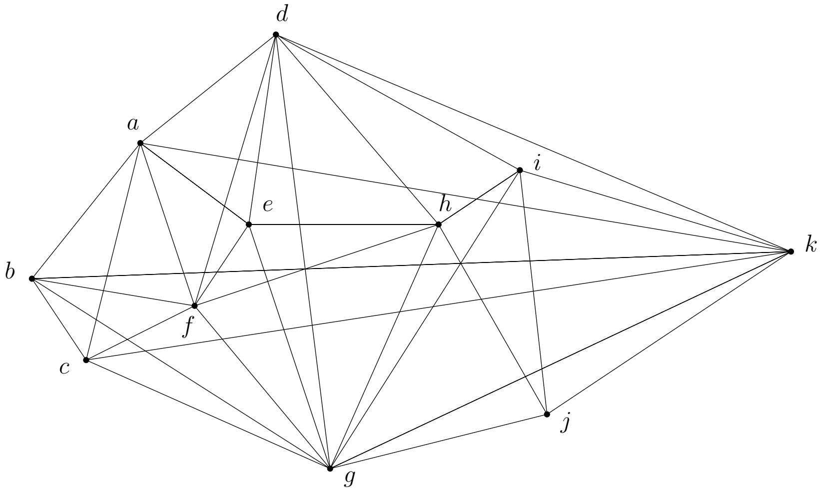


Edge bloc: bounded depth bi-cographs

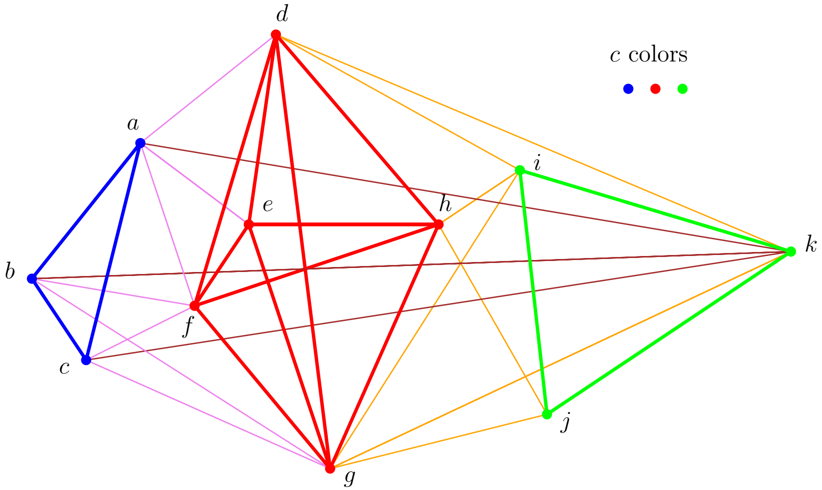




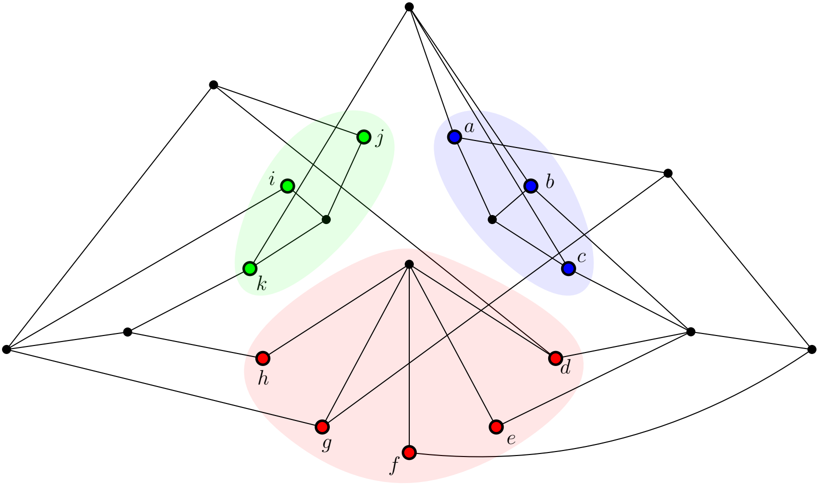
(c, d)-fold coloring



(c, d)-fold coloring



Sparsification: Cut & Paste



Structural Sparsity

Theorem (Gajarský, Kreutzer, Kwon, Nešetril, POM, Pilipczuk, Siebertz, Toruńczyk 2018)

For a class of graphs $\mathscr{C}$ with (c, d)-fold coloring the following are equivalent:

- $\mathscr{C}$ has SC-depth decompositions
- Sparsify($\mathscr{C}$) has tree-depth decompositions;
- Sparsify($\mathscr{C}$) has bounded expansion.
- $\mathscr{C}$ has structurally bounded expansion;

If (c, d)-fold colorings can be computed in time $F(n)$ for $G \in \mathscr{C}$ then checking a first-order sentence ϕ on $\mathscr{C}$ can be done in time

$$F(n) + C(\phi, \mathscr{C})n.$$

Next Step?

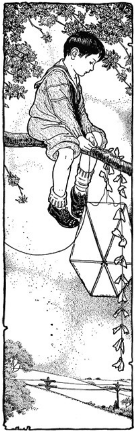
Next Step

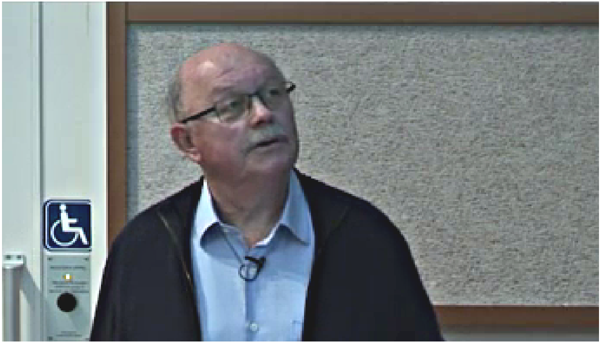
Problem

Let $\mathcal{C}$ be a structurally bounded expansion class.

Is it possible to compute an (c, f)-fold coloring for graphs in $\mathcal{C}$ in polynomial time?

- Would have quite a few algorithmic consequences!





Thank you for your attention.