$(a : b)$-choosability & fractional colourings

Colloque pour Michel

11 octobre 2018

Colouring

An *a*-colouring of a graph G is a mapping $c\colon V(G) \to \mathbf{N}$ such that

- $c(u) \neq c(v)$ whenever $\{u, v\}$ is an edge of G; and
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What about giving more than one colour to every vertex? A colour shall not be assigned to two adjacent vertices.

Set colouring

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- An $(a : 1)$-colouring is simply an a-colouring.
- So $\chi(G)$ is the least integer a such that G admits an $(a : 1)$-colouring.

Naturally

Remark

If G has an $(a : b)$-colouring, then G has an $(am : bm)$-colouring for every $m \geq 1$.

It suffices to "duplicate" each colour m times.

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Let $L\colon V(G) \to 2^{\mathbf{N}}$. An *L-colouring* of G is a colouring c such that $c(v) \in L(v)$ for every vertex v.

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A graph G is a-choosable if it has an L-colouring as soon as $|L(v)| \geq a$ for every vertex v.

The choice number $\mathrm{ch}(G)$ of G is the least integer a such that G is a-choosable.

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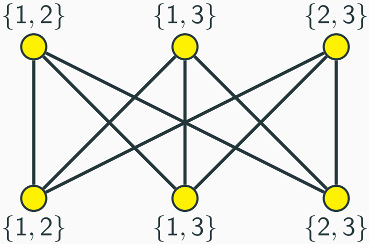

List colouring (2)

Naturally, Erdős, Rubin & Taylor also introduced a list version for set colouring.

A graph G is $(a : b)$-choosable if for every $L\colon V(G) \to 2^{\mathbf{N}}$ satisfying $|L(v)| \geq a$ for every v, there is a set colouring c such that

- $c(v) \subseteq L(v)$ and
- $|c(v)| \geq b$ for every v.

Question (Erdős, Rubin & Taylor, 1979)

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Question

If G is $(a : b)$-choosable, must it be also $(am : bm)$-choosable for every $m \geq 1$?

A first result

Theorem (Tuza & Voigt, 1996)

If G is $(2:1)$-choosable, then G is $(2m:m)$-choosable for every $m \geq 1$.

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The proof relies on a structural characterisation of 2-choosable graphs.

Theorem (Erdős, Rubin & Taylor, 1979)

A connected graph G is 2-choosable if and only if its heart is a single vertex, or an even cycle, or $\Theta_{2,2,2m}$ for some $m \geq 1$.

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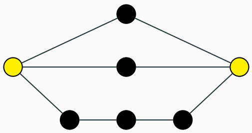


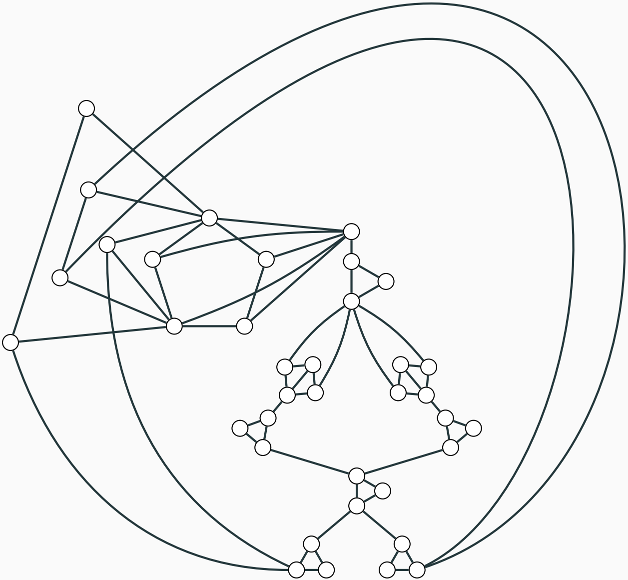
Figure 2: $\theta_{2,2,4}$

An answer (with Z. Dvořák and X. Hu, 2018)...

Theorem

For every $a \geq 4$, there exists a graph G_a that is a-choosable and yet not $(2a : 2)$-choosable.

(4 : 1)-**choosable yet not** (8 : 2)-**choosable**



... leaving a question

What for 3-choosable graphs? Is there a 3-choosable graph that is not $(6:2)$-choosable?

A related question (Erdős, Rubin & Taylor, 1979)

Question

If G is $(a : b)$-choosable, is it $(c : d)$-choosable as soon as $\frac{c}{d} > \frac{a}{b}$?

Colourings

Write $\text{ch}_{:b}(G)$ be the least integer a such that G is $(a : b)$-choosable.

Theorem (Gütner & Tarsi, 2009)

For every graph G and every $\varepsilon > 0$, there exists b_0 such that if $b \geq b_0$, then

$$\text{ch}_{:b}(G) \leq b(\chi(G) + \varepsilon).$$

In other words, if b is lage enough then G is $(b\chi(G) + \lfloor b\varepsilon \rfloor : b)$-choosable.

So?

Corollary

For every $m \geq 3$ and every $\ell > m$, there are G, a and b such that

- G is $(a : b)$-choosable;
- G is not $(\ell : 1)$-choosable; and
- $\frac{a}{b} = m < \ell = \frac{\ell}{1}$.

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3. Apply the theorem to G with $\varepsilon = 1$: there exists b such that G is $(b(\chi(G) + 1) : b)$-choosable.
4. Notice that $b(\chi(G) + 1) = b \cdot m$, and hence setting $a := b \cdot m$ finishes to yield the sought numbers a and b.

Fractional colouring

The fractional chromatic number of G is defined to be

$$\chi_f(G) := \min\left\{\frac{a}{b} \;:\; G \text{ has an } (a:b)\text{-colouring}\right\}.$$

One is naturally tempted to have a list version. Set

$$\mathrm{ch}_f(G) := \inf\left\{\frac{a}{b} \;:\; G \text{ is } (a:b)\text{-choosable}\right\}.$$

Fractional choosability (Alon, Tuza & Voigt, 1997)

Theorem

For every graph G,

$$\mathrm{ch}_f(G) = \chi_f(G).$$

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But is there *anything algorithmic* in your talk?

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Theorem (with Z. Dvořák and J. Volec, 2014)

If G has maximum degree at most 3 and no triangles, then

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Questions

Is there a fixed integer t such that every graph in $\mathcal{C}$ admits a $(14t : 5t)$-colouring?

At least, is there a polynomial time colouring that, given a graph in $\mathcal{C}$, outputs an $(a : b)$-colouring with $\frac{a}{b} \leq \frac{14}{5}$?

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Theorem (Fisher, 1995)

For every integer k, there is a graph G_k with $n := 2^{k+2} - 1$ vertices and $\chi_f(G_k) = \frac{p}{q}$ where

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Take the sequence of Mycielskans of K_3.

Bounding from above

Theorem (Chvátal, Garey & Johson, 1978)

Every n-vertex graph G admits an $(a : b)$-colouring with $\frac{a}{b} = \chi_f(G)$ and $b \leq n^{n/2}$.

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- The complexity of determining χ_f over the class of total graphs is open (*i.e.* colouring both edges *and* vertices).