The induced linkage problem
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Outline

Minor vs induced subgraphs

- A wonderful theory:
  Robertson and Seymour's graph minors
- The most terrible mess:
  excluding induced subgraphs

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What to do ?

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What to do ?

- What to study ?
- What to exclude ?

What to study ?

- Bounding parameters ?
  Tree-width, clique-width, ...
- Optimization ?
  χ, ω, ...
- Recognition problems ?

The induced linkage problem

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- Instance: a graph G, an integer k and k pairs of vertices $(s_1, t_1), \ldots, (s_k, t_k)$
- Question: are there k vertex-disjoint paths $P_1 = s_1 - \cdots - t_1, \ldots, P_k = s_k - \cdots - t_k$

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  Induced = with no edges between them

Interesting:

- in the non-induced version, a corner stone of the graph minor project
- has an obvious parametrization

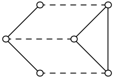
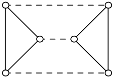
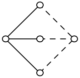
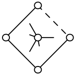
What to exclude ?

- small graphs
  P_3, P_4, claw, bull, triangles, . . .
- "natural" stuff
  paths, cycles, trees
- odd vs even ...
- Stuff that arises from real world problem : interval graphs, O.R., ...

Outline

Truemper configurations

The following graphs are called **Truemper configurations**

- Pyramid: 
- Prism: 
- Theta: 
- Wheel: 

For more about them: see the survey of Vušković.

Original motivation

Theorem (Truemper, 1982)

Let β be a $\{0,1\}$ vector whose entries are in one-to-one correspondence with the chordless cycles of a graph G. Then there exists a subset F of the edge set of G such that $|F \cap C| \equiv \beta_C \pmod 2$ for all chordless cycles C of G, if and only if:
For every induced subgraph G' of G that is a Truemper configuration or K_4 there exists a subset F' of the edge set of G' such that $|F' \cap C| \equiv \beta_C \pmod 2$, for all chordless cycles C of G'.

Our motivation

We are interested in Truemper configurations as **induced subgraphs** of graphs that we study.

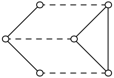
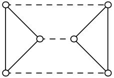
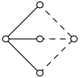
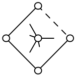
Something classical:

- consider a class of graphs where some Truemper configurations are excluded
- to study a generic graph G from the class, suppose that it contains a Truemper configuration H that is authorized
- prove that $G \setminus H$ must attach to H in a very specific way, so that if H is present, we "understand" the graph.
- continue the study for graphs where H is excluded.
- until so many H's are excluded that again we "understand" the graph

Five classical classes

- Even-hole-free graphs
  (Conforti, Cornuéjols, Kapoor and Vušković 2002)
- Perfect graphs
  (Chudnovsky, Robertson, Seymour and Thomas 2002)
- Claw-free graphs
  (Chudnovsky and Seymour 2005)
- ISK4-free graphs
  (Lévêque, Maffray and NT 2012)
- Bull-free graphs
  (Chudnovsky 2012)

Detecting Truemper configurations

- Pyramid  Polynomial, $O(n^9)$, Chudnovsky and Seymour, 2002
- Prism  NP-complete, Maffray, NT, 2003 Follows from a construction of Bienstock
- Theta  Polynomial, $O(n^{11})$, Chudnovsky and Seymour, 2006
- Wheel  NP-complete, Diot, Tavenas and Trotignon, 2013

Our project

Studying the $2^4 = 16$ classes of graphs obtained by excluding Truemper configurations.

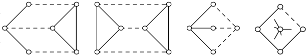
Example : graphs with no prism and no theta, ...

Recognition of the 16 classes

k	theta	pyramid	prism	wheel	Complexity
0	excluded	excluded	excluded	excluded	$O(nm)$
1	excluded	excluded	excluded	—	$O(n^7)$
2	excluded	excluded	—	excluded	$O(n^3m)$
3	excluded	excluded	—	—	$O(n^7)$
4	excluded	—	excluded	excluded	$O(n^4m)$
5	excluded	—	excluded	—	$O(n^{35})$
6	excluded	—	—	excluded	$O(n^4m)$
7	excluded	—	—	—	$O(n^{11})$
8	—	excluded	excluded	excluded	NPC
9	—	excluded	excluded	—	$O(n^5)$
10	—	excluded	—	excluded	NPC
11	—	excluded	—	—	$O(n^9)$
12	—	—	excluded	excluded	NPC
13	—	—	excluded	—	NPC
14	—	—	—	excluded	NPC
15	—	—	—	—	$O(1)$

Chudnovsky, Conforti, Cornuéjols, Diot, Kapadia, Kapoor, Maffray, Seymour, Tavenas, NT, Vušković.

Universally signable graphs (1)

A graph is **universally signable** if it contains no Truemper configuration ().

Examples of such graphs:

- cliques
- chordless cycles
- any graph obtained by gluing two previoulsy built graphs along a clique

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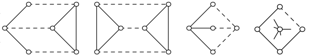
A graph is **universally signable** if it contains no Truemper configuration ().

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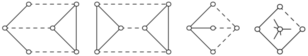
Theorem (Conforti, Cornuéjols, Kapoor and Vušković 1999)

If G is universally signable then G is a clique or G is a chordless cycle, or G has a clique cutset.

Universally signable graphs (2)

Universally signable graphs: no 

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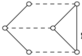
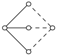
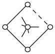
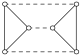
Consequences and open questions:

- Many algorithms (recognition in time $O(nm)$, colouring, max stable set, ...)
- A nice property: every universally signable graph has a **simplicial extreme** (= vertex of degree 2 or whose neighbourhood is a clique).
- Question: recognition in linear time ?

“Only-prism” graphs

A graph is **only prism** if it contains no pyramid , no theta  and no wheel  .

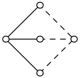
So, the prism  is the only allowed Truemper configuration.

Theorem (Diot, Radovanović, NT and Vušković 2013)

If a graph G is only-prism, then G is the line graph of a triangle-free chordless graph, or G has a clique cutset.

A **chordless** graph is a graph such that every cycle is chordless. The theorem is reversible: any graph obtained by repeatedly gluing line graphs of a triangle-free chordless graphs along cliques is in the class.

Theta-free graphs (1)

Theta: 

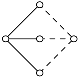
No structural description of theta-free graphs is known so far. But:

Theorem (Chudnovsky and Seymour 2005)

There exits an $O(n^{11})$-time algorithm that decides whether a graph is theta-free.

- Can one be faster?
- Is there a polytime algorithm for computing a max stable set in theta-free graphs?

Theta-free graphs (2)

Theta: 

Theorem (Kühn and Osthus 2004)

There exists a function f such that every theta-free graph G satisfies $\chi(G) \leq f(\omega(G))$.

- the existence of f in the theorem above is non-trivial, for many classes of graphs there is no such f.
- could the function f in the theorem above be a polynomial? A quadratic function?

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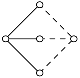
Theta: 

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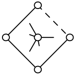
Theorem (Radovanović and Vušković 2010)

If f^* be the smallest possible function in the theorem above, then $f^*(2) = 3$. Rephrased: every $\{$theta, triangle$\}$-free graph is 3-colourable.

Wheel-free graphs

Little is known about wheel-free graphs. Wheel: 

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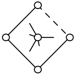
Little is known about wheel-free graphs. Wheel: 

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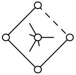
Little is known about wheel-free graphs. Wheel: 

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Theorem (Chudnovsky 2012)

If G is a wheel-free graph, then G contains a multisimplicial vertex (= a vertex whose neighborhood is a disjoint union of cliques).

What about ILP ?

- It is open in the classes above that are NPC to recognize. In particular : NPC for wheel-free graphs.
- It is easily polynomial for the Universally Signable graphs
- It is open otherwise
- In particular it is open for theta-free graphs

Outline

When k is part of the input

The induced linkage problem:

- Instance: a graph G, an integer k and k pairs of vertices $(s_1, t_1), \dots, (s_k, t_k)$
- Question: are there k vertex-disjoint paths $P_1 = s_1 - \cdots - t_1, \dots, P_k = s_k - \cdots - t_k$
  Induced = with no edges between them

NP-complete (Karp, 1972).

Even the non-induced version, even in line graphs

When k is fixed

- Robertson and Seymour: the linkage problem can be solved in time $O(n^3)$
- The induced linkage problem is NP-complete whenever $k \geq 2$

Theorem (Bienstock, 1990)

The following problem is NP-complete.
Instance: G and two vertices a, b
Question: Is there a chordless cycle through a and b ?

Claw-free graphs

Theorem (Robertson, Seymour)

For a fixed k, the induced linkage problem is polynomial when restricted to line graphs (in fact it is FPT).

Theorem (Fiala, Kamiński, Lidický, Paulusma, 2012)

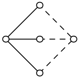
For a fixed k, the induced linkage problem is polynomial when restricted to claw free graph

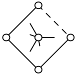
Theorem (Golovach, Paulusma, van Leeuwen)

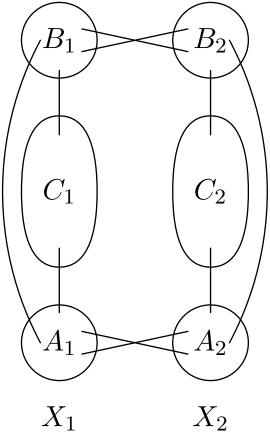
The induced linkage problem is FPT when restricted to claw free graph

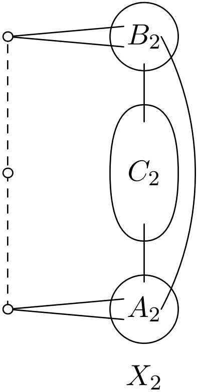
Outline

2-join

Theta: 

Wheel: 





P-graphs

A P-graph is a graph built as follows :

- Pick the line graph of a tree T
- Pick a clique K
- Every vertex in K is adjacent to a vertex in $L(T)$ corresponding to pending edges of T.

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11 axioms in the definition of T...

The main theorem

Theorem (Radovanović, NT, Vušković)

If G is (theta, wheel)-free, then G is a line graph of a triangle-free chordless graph or a P-graph, or G has a clique cutset or a 2-join.

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- First theorem with clique-cutset and 2-join only...
- Consequence : detecting a theta or a wheel in time $O(n^4 m)$.
- Algorithms: clique in $O(n^2 m)$, stable set in $O(n^6 m)$, coloring $O(n^5 m)$

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- Algorithms: clique in $O(n^2m)$, stable set in $O(n^6m)$, coloring $O(n^5m)$
  Induced linkage for fixed k is polynomial

The induced linkage problem in (theta, wheel)-free graphs

- Divide and conquer approach
- The basic class is very easy
- The 2-join : a bottle-neck ...
  We use Charbit, Habib, NT, Vušković algorithm that detects 2-join in time $O(n^2 m)$
- The clique cutset : much harder to use than expected.
  The reason why we do not get FPT as in claw-free graphs ...